Calhoun: The NPS Institutional Archive
DSpace Repository

Theses and Dissertations 1. Thesis and Dissertation Collection, all items

1960

An exact method of servomechanism compensation using S-plane concepts, and an analysis of the effect of passive networks upon steady state performance.

Pollak, Charles D.

Monterey, California: U.S. Naval Postgraduate School

http://hdl.handle.net/10945/12137

Downloaded from NPS Archive: Calhoun

Calhoun is the Naval Postgraduate School's public access digital repository for research materials and institutional publications created by the NPS community. Calhoun is named for Professor of Mathematics Guy K. Calhoun, NPS's first appointed -- and published -- scholarly author.

Dudley Knox Library / Naval Postgraduate School
411 Dyer Road / 1 University Circle
Monterey, California USA 93943

http://www.nps.edu/library

NPS ARCHIVE
1960
POLLAK, C.

AN EXACT METHOD OF SERVOMECHANISM COMPENSATION USING S-PLANE CONCEPTS, AND AN ANALYSIS OF THE EFFECT OF PASSIVE NETWORKS UPON STEADY STATE PERFORMANCE

CHARLES D. POLLAK

DUDLEY KNOX LIBRARY
NAVAL POSTGRADUATE SCHOOL
MONTEREY CA 93943-5101

Library
U. S. Naval Postgraduate School
Monterey, California

A STEADY STATE THEORY OF SERVOMECHANISM COMPENSATION
USING S-PLANE CONCEPTS, AND AN ANALYSIS OF
THE EFFECT OF PASSIVE NETWORKS UPON STEADY STATE PERFORMANCE

* * * * * * * * *

Charles D. Pollak

A RAPID METHOD OF SERVOMECHANISM COMPENSATION
USING S-PLANE CONCEPTS, AND AN ANALYSIS OF
THE EFFECT OF PASSIVE NETWORKS UPON STEADY STATE PERFORMANCE

by

Charles D. Pollak

Lieutenant, United States Navy

Submitted in partial fulfillment of
the requirements for the degree of

MASTER OF SCIENCE
IN
ELECTRICAL ENGINEERING

United States Naval Postgraduate School
Monterey, California

1 9 6 0

NPS Archive
1968
Pollack, C.

P618.5

Library
U. S. Naval Postgraduate School
Monterey, California

AN EXACT [illegible]
[illegible]
THE [illegible] OF PASSIVE NETWORKS [illegible]

by

Charles D. Pollak

This work is accepted as fulfilling
the thesis requirements for the degree of
MASTER OF SCIENCE
IN
ELECTRICAL ENGINEERING
from the

United States Naval Postgraduate School

ABSTRACT

The poles and zeros of a compensator affect the s-plane gain and phase of the open loop system at every point in the s-plane. These effects are studied for poles and zeros on the negative real axis, and a family of curves summarizes the results. A design technique is developed which permits compensation design to satisfy simultaneous specifications of root location and system gain. The method clearly defines the minimum number of compensator sections required, and leads to a logical interpretation of relative needs for phase lead and phase lag compensators.

The writer wishes to express his appreciation for the advice and encouragement given him by Professor George J. Thaler of the U.S.Naval Postgraduate School.

TABLE OF CONTENTS

Chapter	Title	Page
I	Statement of the Problem	1
II	The Root Locus Concept	3
III	The Compensation Procedure	13
IV	Effects of Passive Compensator Networks On The Error Coefficients	25
V	Selected Examples	44
VI	Conclusions	61
VII	Bibliography	62

LIST OF ILLUSTRATIONS

Figure		Page
1.	Basic Block Diagram For Feedback Control System	4
2.	Vector Representation On The S-Plane Of The Function KG(s)	7
3.	A Typical Configuration Of Root Locations On The S-Plane	10
4.	Lead Compensator	15
5.	Determination Of Kv	17
6a.	Example III-1 Root Locus, Uncompensated System	21
6b.	Example III-1 Root Locus, Compensated System	22
7.	Example III-2 Root Locus, Compensated System	24
8.	S-Plane Diagram, Lead Compensator Section	30
9.	Compensator Ratio And Phase Angle Change vs. Negative Real Axis Location	32
10.	S-Plane Diagram, Second Order Uncompensated System	37
11a.	Example V-1, Desired Root Location	45
11b.	Example V-1, Root Locus, Compensated System	49
12a.	Example V-2, Root Locus, Compensated System	52
12b.	Example V-2, Transient Response, Compensated System	53
13.	Example V-3, Root Locus, Compensated System	55
14.	Example V-4, Root Locus, Compensated System	60

DEFINITIONS OF SYMBOLS

s = complex variable = $\sigma \pm j\omega$

s = point on s-plane selected to be a root, r .

P = pole coordinate

Z = zero coordinate

Zd = distance from a zero to a root location

Pd = distance from a pole to a root location

ζ = damping ratio = $\cos \tan^{-1} \omega r/\sigma r$

r = $\sigma r \pm j\omega r$ = root location

Zm = location of compensator zero to produce a maximum value for Z/Zd

ωn = $\sqrt{\sigma^2 r + \omega^2 r}$

ωnc = value of ωn after compensation

Z_L = location of zero which just satisfies angle criterion when pole is at infinity

CHAPTER I. Statement of the Problem.

Compensators are required in feedback control systems to make possible the simultaneous satisfaction of static, steady state, and dynamic performance specifications. Most compensator design procedures are based upon manipulation of the open loop transfer function. The specific manipulations usually involve trial and error techniques, and graphical aids are popular since they speed up the process of converging to an acceptable compensator design. Bode diagram[1] methods set the gain to a value which is adequate for static and steady state accuracy specifications, then permit adjustment of dynamic performance by manipulation of the magnitude and phase curves to obtain acceptable phase and gain margains. Polar plot[1] and Nichols Chart[1] design techniques also set the gain to a desired value, and adjust the open loop transfer function curve to establish the resonance peak and resonant frequency of the closed loop frequency response curve. The root locus method,[2,3,4] however, deals directly with dynamic performance by concerning itself with root locations on the s-plane. Compensation is designed by introducing poles and zeros which force the root locus to pass through a preselected point, and the gain is adjusted to place the root at this point. Since an infinite number of pole and zero locations can produce this result, each with a different

gain value, trial and error is required. It has been[5] shown that compensators can be designed to place complex roots at selected points with a specified gain value and without trial and error, but requires that the gain be specified precisely.

The method developed in this paper solves the compensation problem using a simplified trial and error method. Equations are developed which permit preliminary estimates of both gain and phase effects on the s-plane, and a set of curves is also provided for use in both preliminary estimates and final calculations. As a result of using these tools the initial trial is frequently quite close to the desired answer. The method is applicable to any order system.

CHAPTER II. The Root Locus Concept.

There is in the literature detailed information concerning the theory, significance, and application of s-plane techniques and, specifically, the root locus method[1,2,3,4]. No attempt can be made here to duplicate this material, but a basic review of the theory is appropriate.

The Laplace transfer function of a feedback control system may be utilized to relate the output to the input of the system as follows (see fig. 1):

$$\frac{\theta c}{\theta r}(s) = \frac{KG(s)}{1 + KGAH(s)} , \qquad (2\text{-}1)$$

where $\theta c(s)$ refers to the system output,

$\theta r(s)$ refers to the signal input,

$KG(s)$ is the transfer function of the forward loop,

$AH(s)$ is the transfer function of the feedback path.

$KG(s)$ and $AH(s)$ are rational polynomials in s .

The characteristic equation is obtained when the denominator of $\frac{\theta c(s)}{\theta r}$ vanishes; i.e., when

$$1 + KGAH(s) = 0 \quad . \qquad (2\text{-}2)$$

For systems employing unity feedback, $AH(s) = 1$, and the characteristic equation becomes simply

$$1 + KG(s) = 0 \qquad (2\text{-}3)$$

To simply the dicussion, unity feedback is assumed throughout the remainder of the paper unless otherwise

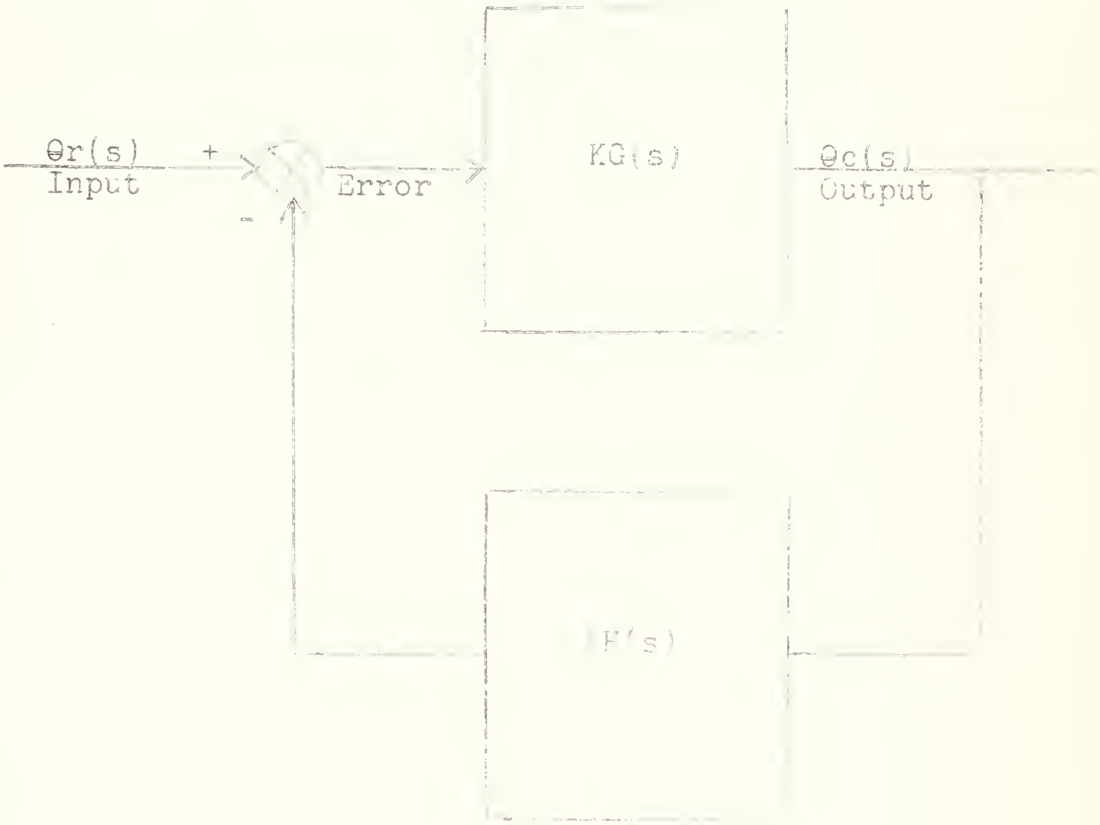


FIG 1

BASIC BLOCK DIAGRAM FOR FEEDBACK CONTROL SYSTEM

stated.

The roots of the characteristic equation are those values of s which satisfy the equation $KG(s) = -1$. Values of s represent complex numbers which may be plotted on the complex plane $s = \sigma + j\omega$. Therefore values of s which satisfy the characteristic equation are located on the s-plane as individual points.

Any location $s = \sigma + j\omega$ may be represented as a vector quantity having magnitude and phase angle, in which the vector originates from the origin and in which the phase angle is measured counterclockwise from the positive σ axis. The quantity -1 may be treated as a unit vector with a phase angle of 180°.

Suppose we have a transfer function of the form

$$F(s) = \frac{K(s+Z_1)\ldots}{s(s+P_1)(s+P_2)\ldots}$$

Values of s which make the denominator identically zero are called poles and the numerator values are called zeros. These may be plotted on the s-plane and are commonly denoted by the symbols "X" for pole and "O" for zero.

For any arbitrary point S , a term such as $s+P_1$ is then a vector quantity. All of the factors associated with F(s) , except the scaler gain number K , similarly may be treated as vector quantities. The product

of these vectors is also a vector, whose magnitude is the product of the individual vector magnitudes and whose phase angle is the sum of the individual angles. For convenience, the reference point for the vectors is taken from each pole and zero in turn. Pole angles are then positive and zero angles negative. See fig. 2.

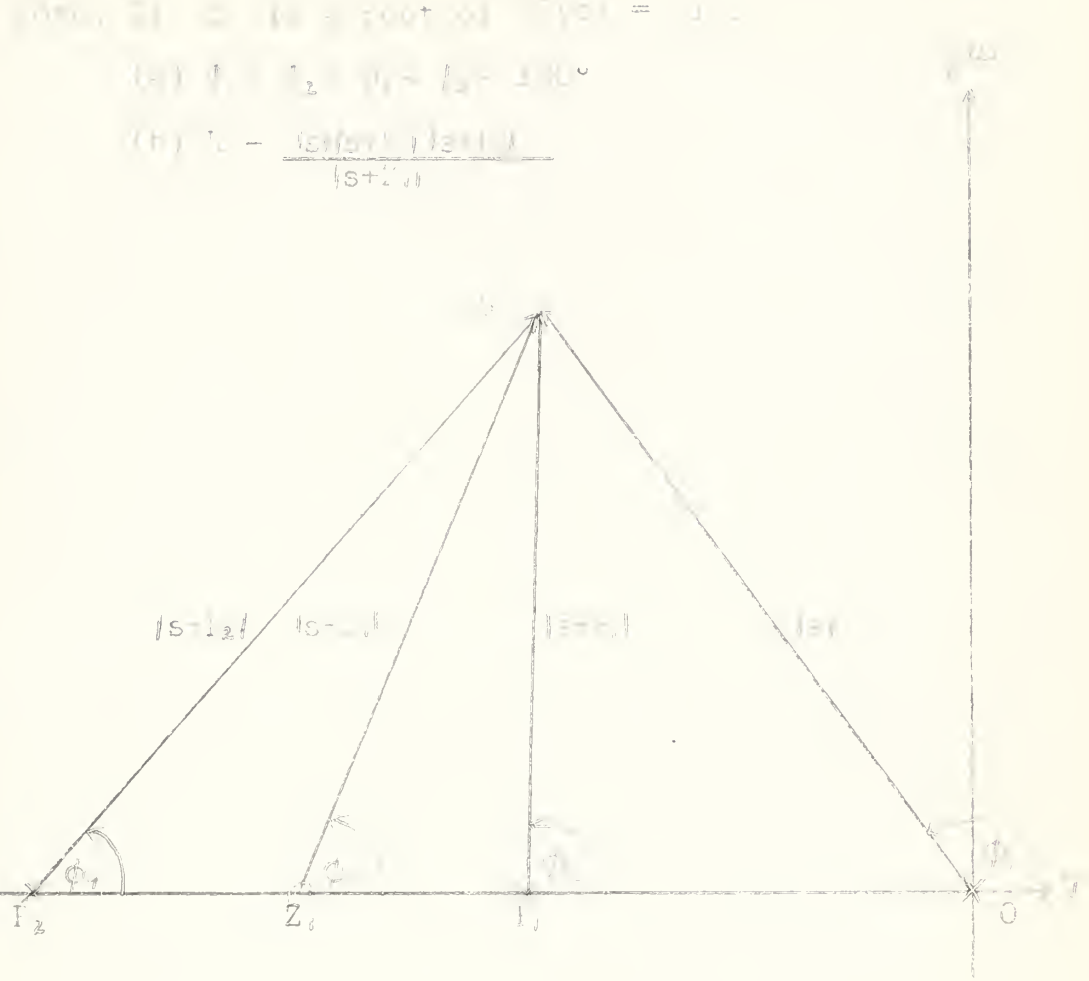
Since F(s) = -1 defines the characteristic equation of the system, only those values of s are roots which, when substituted into the equation, produce a unit vector at 180°. The two requirements of phase angle and magnitude commonly are separated. The poles and zeros are first plotted on the s-plane. Then a locus of points is determined which satifies the phase angle requirements of 180° (or odd multiple thereof *). This locus is termed the root locus, since it is the locus of all possible locations of the roots depending upon the value associated with K . Finally the particular values of roots are located by rearranging the characteristic equation into the form

$$K = \frac{|s|\,|s+P_1|\,|s+P_2|\ \ldots}{|s+Z_1|\ \ldots} \qquad (2\text{-}4)$$

and locating values by trial and error which satisfy the relationship along the root locus.

---

*Note that the quqntity (-1) may be represented as $1\angle 180^\circ$, $1\angle 540^\circ$, $1\angle 900^\circ$, etc.



FIG

VECTOR REPRESENTATION ON THE S-PLANE
OF THE FUNCTION KG(s)

While this procedure may at first appear to be rather laborious, it is in fact quite easy. Certain thumb-rules are given in the literature which specify the number of root locus segments, the method for finding asymptotes of the root locus curves, general shapes of the curves to be expected, etc. In addition, the location of the root locus is facilitated greatly by the use of a device for adding and subtracting the phase angles. Particularly recommended is the Spirule, a simple and inexpensive device which permits rapid location of the root location on the locus and the roots with an accuracy of about 2° in angle and 5% in gain.

There are many advantages in the root locus method. The roots permit the evaluation of the transient and frequency response. Furthermore, since the root locus is by definition the locus of all possible roots depending upon the number associated with the gain factor K , the effect on system performance associated with changes in gain can be immediately determined. Thus the root locus offers perhaps the best single means of system evaluation.

For example, suppose we desire to examine the transient response of a system to a unit step input. This is accomplished by evaluation of the residues at the root locations. $\theta r(s)$ for a unit step is $\frac{1}{s}$, and from the

expression

$$\frac{\theta c}{\theta r}(s) = \frac{KG(s)}{1 + KG(s)} \quad , \qquad (2-5)$$

we obtain

$$\theta c(s) = \frac{1}{s} \cdot \left[\frac{K(s+Z_1)\ldots}{(s+r_1)\ldots} \right] \qquad (2-6)$$

where $r_1, \ldots$ refers to the roots of the characteristic equation. Therefore,

$$\theta c(t) = \mathcal{L}^{-1} \frac{1}{s} \cdot \frac{\prod_{i=1}^{i=j} (s+Zi)}{\prod_{j=1}^{j=n} (s+r\ )} \qquad (2-7)$$

Those roots which are closest to the origin generally dominate the system response. In fig. 3 is shown a typical configuration of roots on the s-plane (roots are commonly designated by "X" as are poles). As a general rule, most systems do not have more than one pair of complex conjugate roots near the origin.

Roots on the negative real axis produce exponentially damped terms in the transient response. Roots on the imaginary $j\omega$ axis produce undamped sinusoidal responses. Complex roots produce a sinusoidal damped response which is characteristic of a well designed servomechanism.

Fig. 3 indicates certain quantities which may be associated with dominant complex roots. ωn is the natural frequency and ωt the transient oscillating frequency associated with

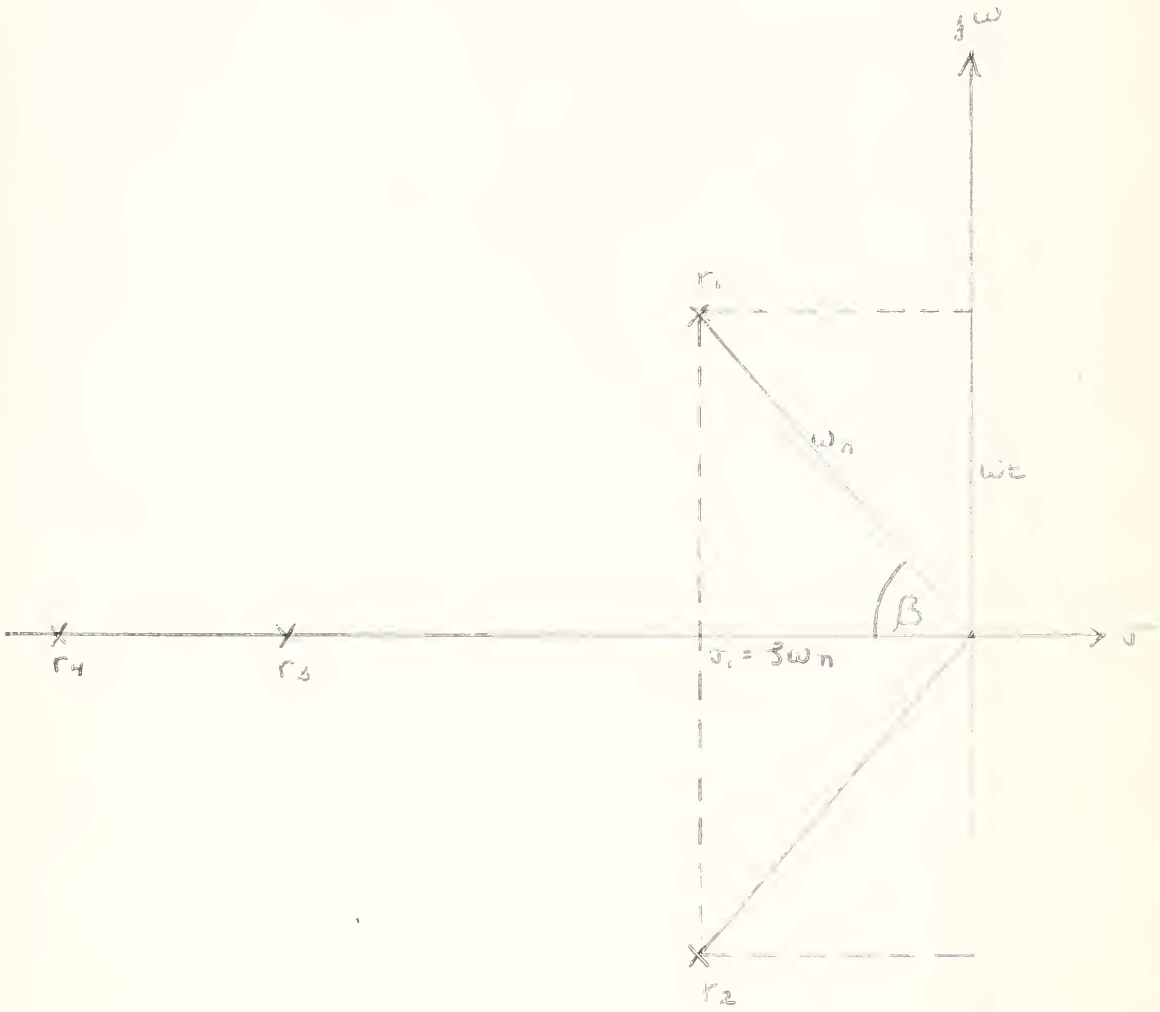


FIG 3

A TYPICAL CONFIGURATION

OF ROOT LOCATIONS ON THE S-PLANE

the dominant pair. β is the angle between the negative real axis and the line connecting the complex roots to the origin. $\cos\beta = \zeta$, the familiar damping ratio. The real part of r_1 and r_2 is $\sigma_1 = \sigma_2 = \zeta\omega_n$. r is located at $\sigma_3 + j0$.

In general, if all roots (other than the complex dominant roots) are at a distance from the origin $\geq 2\omega_n$ they may be neglected in determing the transient response since the time constant and/or coefficient associated with such roots will be small in comparison to the dominant roots. In this case the system behaves essentially as a second order system and the following relationships are valid:

$$\text{settling time} = 4\tau = 4/\zeta\omega_n \tag{2-8}$$

$$\text{number of oscillations} = \frac{2}{\pi}\,\frac{\sqrt{1-\zeta^2}}{\zeta} \tag{2-9}$$

If two complex roots and one real root dominate the response, the system behaves essentially as a third-order system. If four roots contribute significantly to the response, the system is fourth-order, etc. Obviously the estimation of transient performance for higher order systems is more difficult than for second-order systems. It may be necessary in some cases to take the inverse Laplacian. Neverless, the location of the roots generally will permit a "feel" for the response to be expected.

A great deal has been written about compensation in the s-plane. One of the most powerful aspects of the s-plane technique is the flexibility it affords in designing proper compensation. The method may be applied to multiple loop systems as well as to single loop and, in fact, it is precisely in the design and analysis of complex systems that the superiority of the method over other techniques is most apparent.

The general approach in s-plane compensation has been to study the effect of the root locus curves with the addition of poles and zeros introduced by compensating networks. The method proposed herein adopts a somewhat different point of view. This approach is explained in Chapter Three.

CHAPTER III. The Compensation Procedure.

The root locus discussion in the preceeding chapter described in broad terms the response to be expected from system roots located at various places on the s-plane. The designer of a servomecanism must, in general, first approach the problem in the other direction; that is, he must place the roots of the system in such a manner that the system will meet design specifications. Thus there will be a point on the s-plane where roots are desired (or, more practically, there will be an area on the plane anywhere within which the roots will produce an acceptable response). The proper placement of roots will vary widely depending upon the system. In some cases, particularly in multiloop systems, the final location of roots may require adjustment because of additional roots near the complex pair. This is, however, a matter of final system adjustment. For purposes of this discussion, it is presumed that the desired location of the complex dominant roots is known.

It has been shown that the root locus is, by definition, the locus of all possible root locations. Each root locus segment will contain one root. As gain is varied from 0 to ∞ each root will move along its respective locus segment from a pole to a terminating zero (or to ∞ if the segment does not terminate in a finite zero).

This locus, futhermore, is that locus of phase angles measured from the various poles and zeros which is equal to 180°, or add multiple, thereof. If the locus does not pass through the desired root location, compensation networks must be added.* This may be accomplished by:

(a) Rearrangement of existing poles and zeros so as to produce a phase angle of 180° at the desired root location.

(b) The addition of suitably placed poles and zeros so as to produce a phase angle of 180° at the desired root location.

(c) A combination of (a) and (b).

That is, if poles and zeros are rearranged or added so as to produce a phase angle at the desired root location, then by definition that location will lie on the root locus. This does not necessarily produce suitable compensation by itself, since steady state requirements and other considerations must be included as well, but these additional considerations generally may be handled without excessive difficulty.

Consider first a phase lead (derivative) compensator. The transfer function:

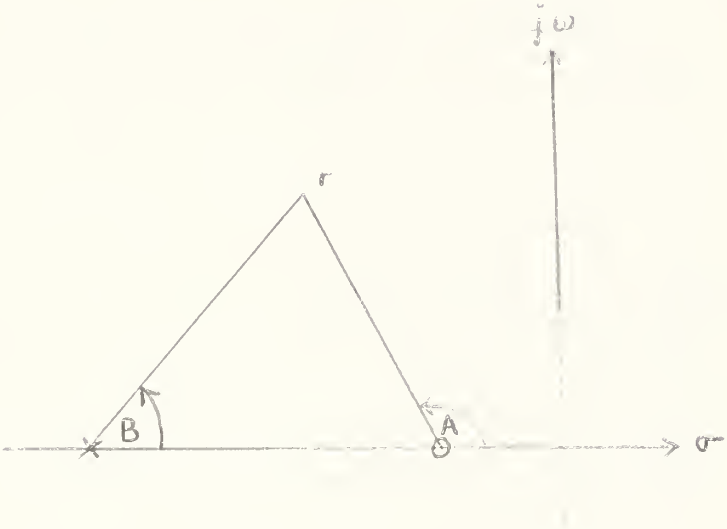
$$T.F. = \alpha \frac{s\tau + 1}{\alpha s\tau + 1} = \frac{s+Z}{s+P},$$

where α is a number less than 1, Z is the zero of the compensator, P is the pole of the compensator.

* In addition, of course, compensating devices may be required to meet system requirements of bandwidth, steady-state accuracy, etc.

Because $\alpha < 1$, $Z < P$, and therefore the zero is closer to the origin than the pole. See fig. 4.

FIG. 4
LEAD COMPENSATOR



For a position r of the dominant roots, $A > B$. Since the phase angle at $r = \sum$ pole angles $- \sum$ zero angles, a lead compensator will decrease the phase angle at r.

For a lag compensator, the transfer function is

$$T.F. = \frac{s\tau+1}{s\tau_1+1} = \frac{s+Z}{s+P}$$

In the case of a phase lag compensator, $Z > P$ and the pole is closer to the origin than the zero. The phase angle contribution at r will be an increase in the total phase angle.

To illustrate the effects of the above concepts, suppose a configuration of poles and zeros is given and a

desired root location is known. Suppose further that a phase angle of 220° is measured at r . Introduction of a phase lead compensator will reduce the phase angle. If a compensator is added which introduces -40°, the phase angle will be 180° and r will lie on a root locus. If the phase angle at r originally was measured to be 170°, a lag network of 10° would place r on the root locus.

One of the most important considerations in a system is steady state accuracy. For a type 1 servomechanism, the steady state accuracy is proportional to the velocity constant Kv . Generally Kv may be increased above specifications but should not be decreased.* In compensating Kv may be increased most readily by means of a phase lag compensator. This is proven in Chapter IV. On the s-plane the value of Kv is quickly determined. If Z_1, Z_2,... are the distances from the origin to the location of zero 1, zero 2,..., and P_1 ,P_2 ,... are the distances from the origin to the location of pole 1,

---

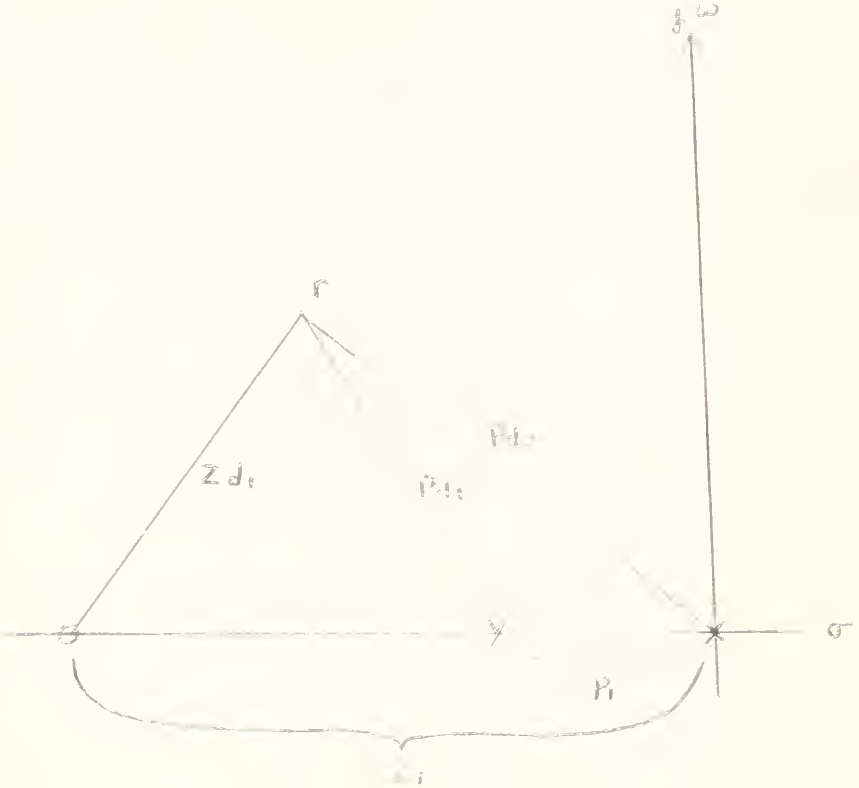
* However, since Kv is a measure of the amplification requirements of the system, it may not be desirable to increase Kv beyond the value specified.

pole 2,... and if $Zd_1, Zd_2, \ldots$ are the distances from zero 1 to the root location r etc., and $Pd_1, Pd_2, \ldots$ are the distances from pole 1 to r etc., then

$$Kv = Pd_0 \frac{Pd_1 \; Pd_2 \; \ldots Z_1 \; Z_2 \; \ldots}{P_1 \; P_2 \; \ldots Zd_1 \, Zd_2 \ldots} \qquad (3\text{-}1)$$

See fig. 5.

FIG. 5
DETERMINATION OF Kv



This may be shown to be true as follows:

$$\text{The root locus gain } K = \frac{\text{product of pole distances}}{\text{product of zero distances}}$$

$$= \frac{Pd_1 \, Pd_2 \ldots}{Zd_1 \, Zd_2 \ldots}$$

$$G(s) = \frac{K(s+Z_1)(s+Z_2)\ldots}{s(s+P_1)(s+P_2)\ldots} = \frac{Kv(s\tau a+1)(s\tau b+1)\ldots}{s(s\tau_1+1)(s\tau_2+1)\ldots} \qquad (3\text{-}2)$$

where $\tau a = \frac{1}{Z_1}$, $\tau b = \frac{1}{Z_2}$, $\tau_1 = \frac{1}{P_1}$, $\tau_2 = \frac{1}{P_2}$, etc.

To put $(s+Z_1)$ in the $(s\tau a+1)$ form, it is necessary to multiply and divide the numerator of G(s) by Z_1 , giving

$$G(s) = \frac{KZ_1(s\tau a+1)(s+Z_2)\ldots}{s(s+P_1)(s+P_2)\ldots} \qquad (3\text{-}3)$$

Each term may be taken in succession, so that

$$G(s) = \frac{K(s\tau a+1)(s\tau b+1)\ldots Z_1 Z_2}{s(s\tau_1+1)(s\tau_2+1)\ldots P_1 P_2} \qquad (3\text{-}4)$$

But $K = \frac{Pd_1 Pd_2 \ldots}{Zd_1 Zd_2 \ldots}$

$$\therefore G(s) = \frac{Pd_1 Pd_2 \ldots Z_1 Z_2 \ldots (s\tau a+1)(s\tau b+1)\ldots}{P_1 \; P_2 \ldots Zd_1 Zd_2 \ldots s(s\tau_1+1)(s\tau_2+1)\ldots}$$

$$= \frac{Kv(s\tau a+1)(s\tau b+1)\ldots}{s(s\tau_1+1)(s\tau_2+1)\ldots} \qquad (3\text{-}5)$$

The general procedure for compensation is as follows:

(a) Plot the poles and zeros of the transfer function of the uncompensated system on the s-plane, draw the root locus, and locate the roots for the specified Kv (type 1 system will be assumed).

(b) From the specifications, select the desired complex root location r (or area of satisfactory root location).

(c) If the uncompensated system roots are not satisfactory, measure the phase angle at r .

(d) If the phase angle at r is greater than 180°, lead compensation will be necessary. If necessary,

adjust Kv at r by use of a lag network (in some cases it may be possible to adjust Kv with a lead network, as shown in Chapter IV).

(e) Place the poles and zeros of the total compensated system in such a manner that the corrected phase angle at r is 180°.

(f) IF deemed necessary, draw the root locus for the compensated system to locate the position of introduced roots.

In the examples that follow, the following premises are made:

(a) A desired root location r is known.

(b) The attenuation ratio of phase lead compensators should not exceed 10.*

(c) The specified Kv of the compensated system should not be reduced below the value given, but may be increased if necessary.

---

*A practical consideration in the case of lead compensators is that high attenuation ratios produce excessive noise in the system. Therefore an attenuation ratio

$$\frac{P}{Z} > 10$$

generally are avoided.

Example III-1

$$G = \frac{45}{s(s+10)(s+1)}$$

Required:

For compensated system, $K_v \geq 2$

$\omega_n = 2$, $\zeta = 0.7$

Solution: See fig 6a

The root locus is drawn with the aid of a spirule and the root locations are determined to be $s = -.25 \pm j2.0$, and $s = -10.5$. The desired root is located at $\zeta = 0.7$, $\omega_n = 2$. The phase angle at this location is measured and found to be 250°, and therefore phase lead compensation is required. In this case, pole cancellation is attractive in order to remove the pole at $s = -1$. Therefore a zero is placed at $s = -1$ and the required P location to produce a phase angle of 180° at the desired root location is found by spirule to be at $s = -3.2$. The attenuation ratio of the compensator = $\frac{3.2}{1} = 3.2$.

$$K = (2)(2.2)(8.7) = 42$$

$$K_v = \frac{42}{(2.2)(8.7)} = 2$$

$$G(s) = \frac{42}{s(s+1)(s+10)} \cdot \frac{(s+1)}{(s+3.2)}$$

The root locus for the compensated system is shown in fig. 6b.

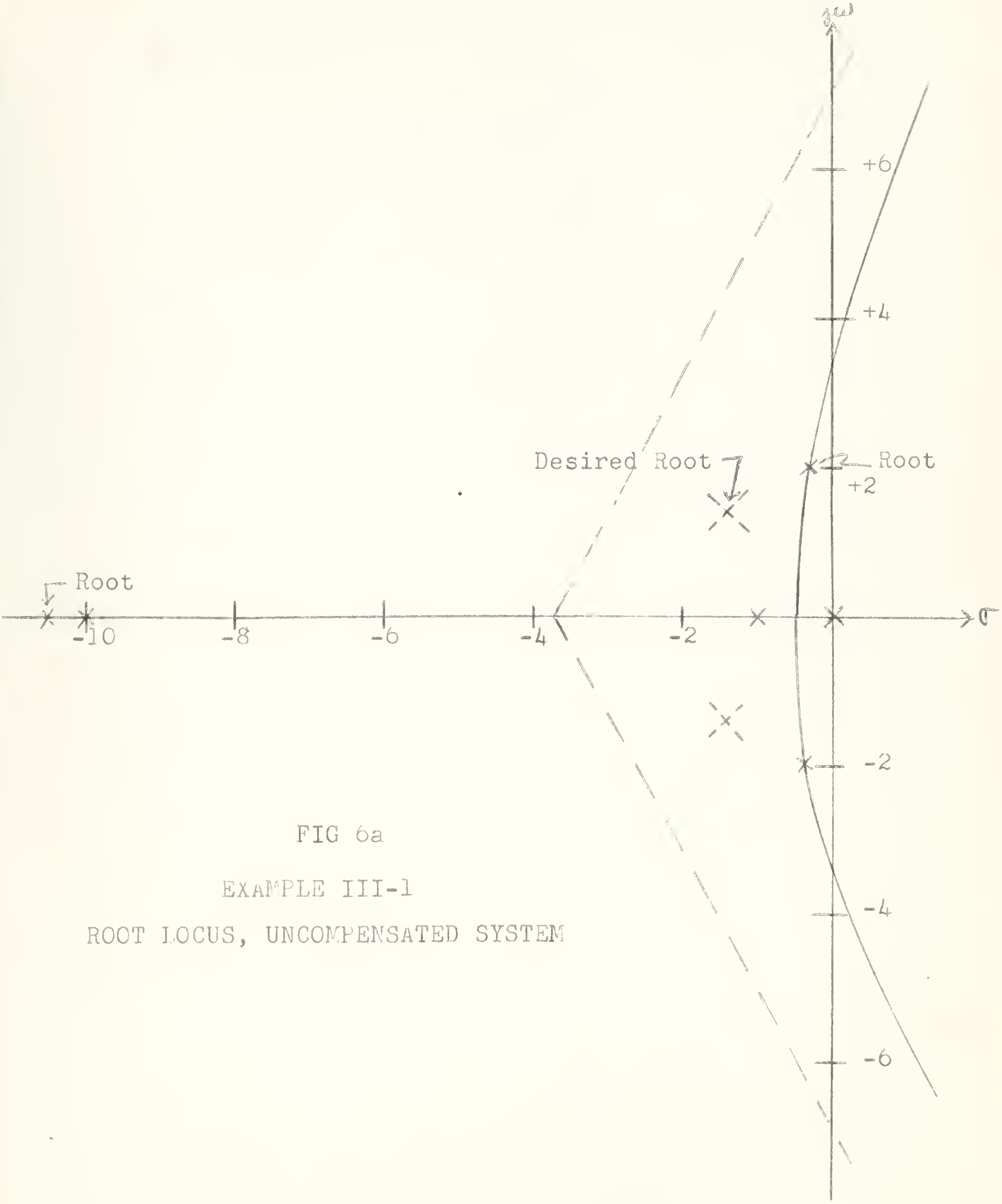


FIG 6a

EXAMPLE III-1

ROOT LOCUS, UNCOMPENSATED SYSTEM

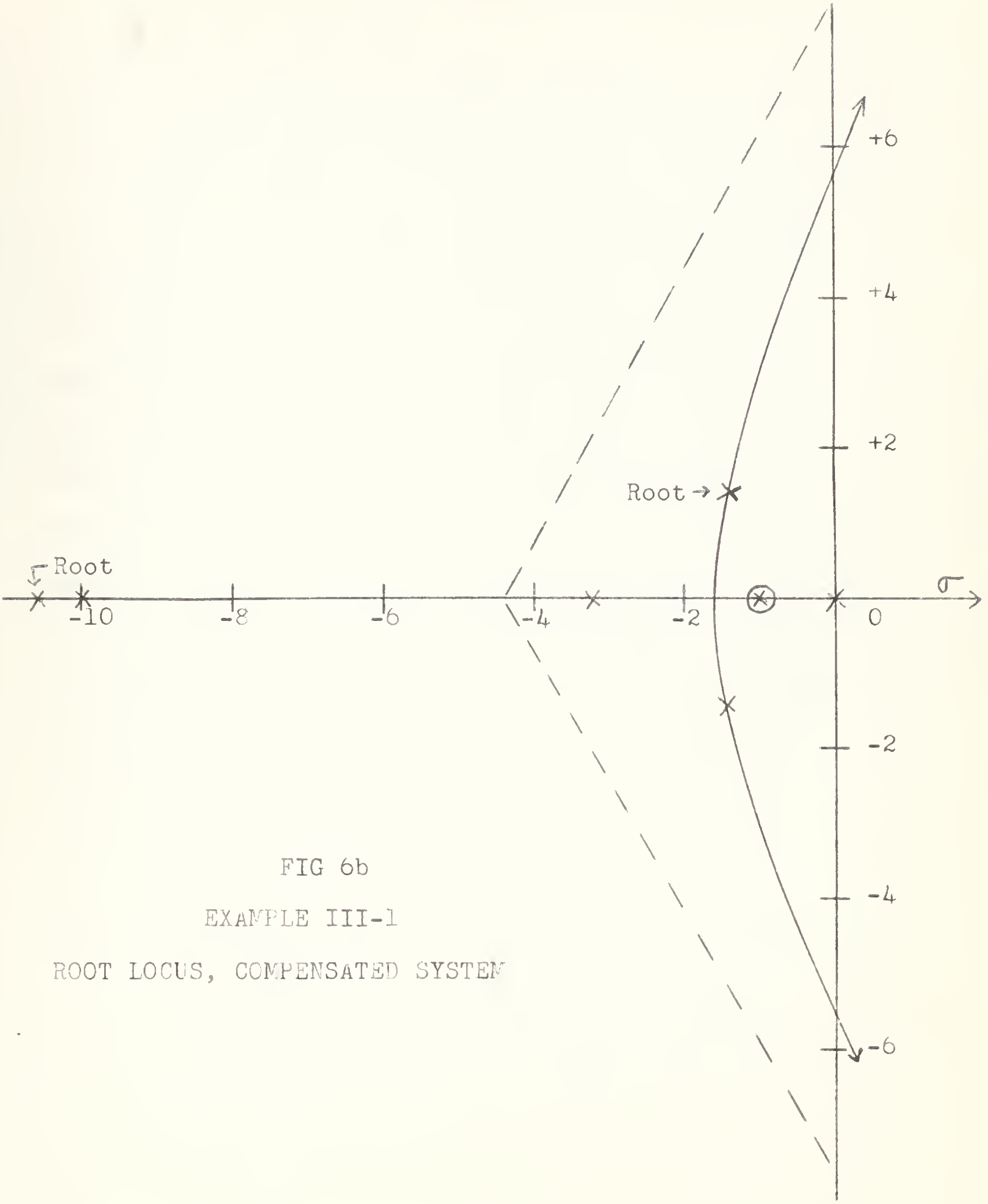


FIG 6b

EXAMPLE III-1

ROOT LOCUS, COMPENSATED SYSTEM

Example III-2

Same as the previous example except that Kv = 6.

Solution: (See fig. 7)

In this case it does not appear warranted to attempt to alter the pole and zero location of the phase lead compensator. This is due to the fact that the zero was chosen to cancel the pole effect at s= -1, and also because of the fact that Kv must be increased by a factor of 3. However, from root locus studies, it is apparent that as long as the zero remains between the pole at s= -1 and s= -10, then the root location will be part of the segment originating between the two poles at the origin and at s= -1. The other root locus segment will lie along the real axis between the zero and the pole at s= -10 (a mental analysis of this type is desirable because it will generally indicate whether or not an undesirable root may be produced by the introduction of a compensator or even an unstable root placed in the right half plane).

Since Kv must be raised by a factor of 3, $\frac{Z}{P}$ of the lag network should be ≥ 3. By placing a dipole near the origin, the phase angle will be affected only slightly. For illustration purposes, select P=0.2, Z=0.8. The zero of the phase lead compensator was selected to be

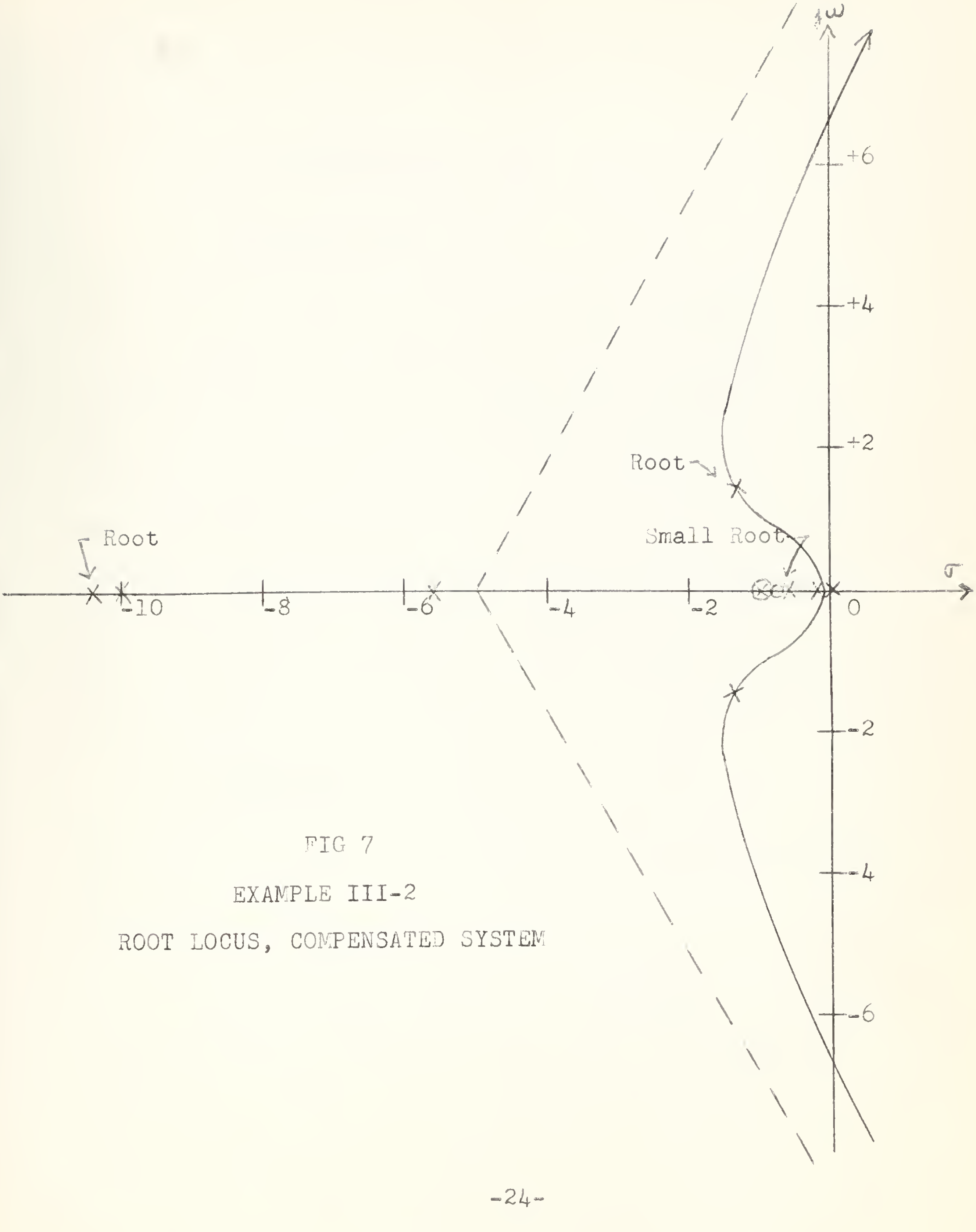


FIG 7

EXAMPLE III-2

ROOT LOCUS, COMPENSATED SYSTEM

at s= -1 . The pole of the phase lead compensator is placed in a "corrected" position to produce a phase angle of 180° at the desired root location. The position is at s= -5.6 . Then

$$K = \frac{2(1.9)(4.5)(8.7)}{1.6} = 93$$

$$Kv = \frac{93(0.80)}{(0.2)(5.6)(10)} = 6.6$$

It will be noticed that the phase lag compensator ratio did not raise Kv by a factor of 4, but by a factor somewhat less. This occurs because, in adjusting the pole of the phase lead compensator to achieve a phase angle of 180° (after placing the pole and zero of the lag compensator), the lead compensator pole moved to the left. This will increase K slightly, since the pole is farther from the root location, but it will have a larger effect proportionately in the determination of

$$Kv = \frac{KZ_1 Z_2 \ldots}{P_1 P_2 \ldots}$$

This effect will be explained more fully in the following chapter.

Final compensation:

$$G = \frac{93}{s(s+1)(s+10)} \cdot \frac{(s+1)}{(s+5.6)} \cdot \frac{(s+0.8)}{(s+0.2)}$$

The root locus of the compensated system is shown in fig. 7 .

CHAPTER IV. Effects of Passive Compensator Networks On The Error Coefficients.

Compensation methods using passive networks have been developed by several authors.[2,3,4,5] In general mathematical formulae have been employed which express a relationship between a pole position and the corresponding zero location necessary to place the desired root location r on the root locus. It then becomes necessary to select arbitrarily either the pole or zero location and the other may be calculated. In this respect there is no difference between earlier methods and that procedure described in Chapter III.

Heretofore no guides have established which indicate the optimum position at which to place either the zero or the pole in order to achieve maximum effect from the compensator section. In fact it was not known whether optimum positions existed. Therefore an investigation was undertaken to study the effects of pole-zero location upon the system.

Because of the importance of steady state accuracy in most servomechanism applications, the effect upon steady state error of different compensator configurations was considered to be of primary importance. Simply stated, the problem is this: a compensator is needed to place

the root locus through a desired position r . Assuming that a single (or a specified number of sections) is sufficient to accomplish this, there nevertheless remains an infinite number of possible pole-zero combinations for the section which will produce a phase angle of 180° at r . It then becomes advantageous to know the effect upon system performance of different configurations and, specifically, upon steady state accuracy.

As a result of Chu's[6] studies in phase angle loci and gain loci on the s-plane, it may be stated that any open loop transfer function produces a phase angle and a gain value at every point on the s-plane. These numbers depend only on the locations of the open loop zeros and poles, not on the gain constant of any associated amplifying equipment. Certain lines on the s-plane are root loci, i.e., loci of possible roots for the closed loop system. All points on such loci have a phase angle which is an odd multiple of π . Each point on the root loci has a specific gain number associated with it. A root of the closed loop system may be moved to a selected point on a root locus by adjusting the system amplication constant until the necessary gain number obtains.

For any point, s_1 , not on a root locus the phase angle is $\underline{|G(s=s\ )}$ and the gain number is

$$K = \frac{|s| \prod_{i=1}^{N} |s - P_i|}{\prod_{i=1}^{M} |s - Z_i|} \tag{4-1}$$

As shown in the previous chapter, in order to design a compensator such that a select point, s_1, becomes a root, r, it is necessary first of all to introduce poles and zeros such that for the compensated transfer function G_c, the angle at point s_1 becomes $\angle G_c(s = s_1) = (2C-1)\pi$, where C is any positive integer. This merely means that a root locus must be forced to pass through s_1. It is also necessary that the gain be adjusted to move the actual root point along the root locus to s_1. An infinite number of possible pole-zero combinations can satisfy the angle requirements, but for each a different gain number obtains at s_1, and thus for each pole-zero combination a different error coefficient[7] obtains. Since steady state accuracy depends on the error coefficient only certain values are allowable, and it is this condition that results in trial and error design methods.

Assuming that the designer knows the uncompensated $G(s)$, and can select a point $s_1 = r$ at which a root is to be located, it is a simple matter to compute $\angle G(r)$ using either analytic or graphical methods. Assuming also a passive compensator with poles and zeros restricted to the negative real axis, the maximum phase angle producible by a single section is obtained when the pole is at minus infinity and the zero at the origin (or vice versa)

and this maximum phase angle is

$$\beta = \pm \left(\pi - \tan^{-1} \frac{\omega_1}{\sigma_1}\right) \qquad (4\text{-}2)$$

where $r = s_1 = \sigma_r + j\omega_r$. Therefore the minimum number of compensator sections needed to satisfy the angle condition is:

$$\text{MINIMUM NO. OF SECTIONS} = \frac{\pi - \underline{|G(r)}}{\beta} \qquad (4\text{-}3)$$

The gain number, K(r), is readily evaluated, and the error coefficient (which does not exist at $s_1 = r$ because this point is not yet on a root locus but which may be computed formally) is found from

$$Kx \triangleq K(r) \frac{\prod_{i=1}^{N} (Zi)}{\prod_{i=1}^{M} (Pi)} \qquad (4\text{-}4)$$

where $Kx \triangleq$ Kp, Kv, Ka, ... depending on the system type number.

This section considers first the general concepts involved for any type system of any order, then considers in detail the second order Type One system, developing relationships which may be used as a guide to design.

LEAD SECTIONS

The s-plane diagram of fig. 8 may be used to study the

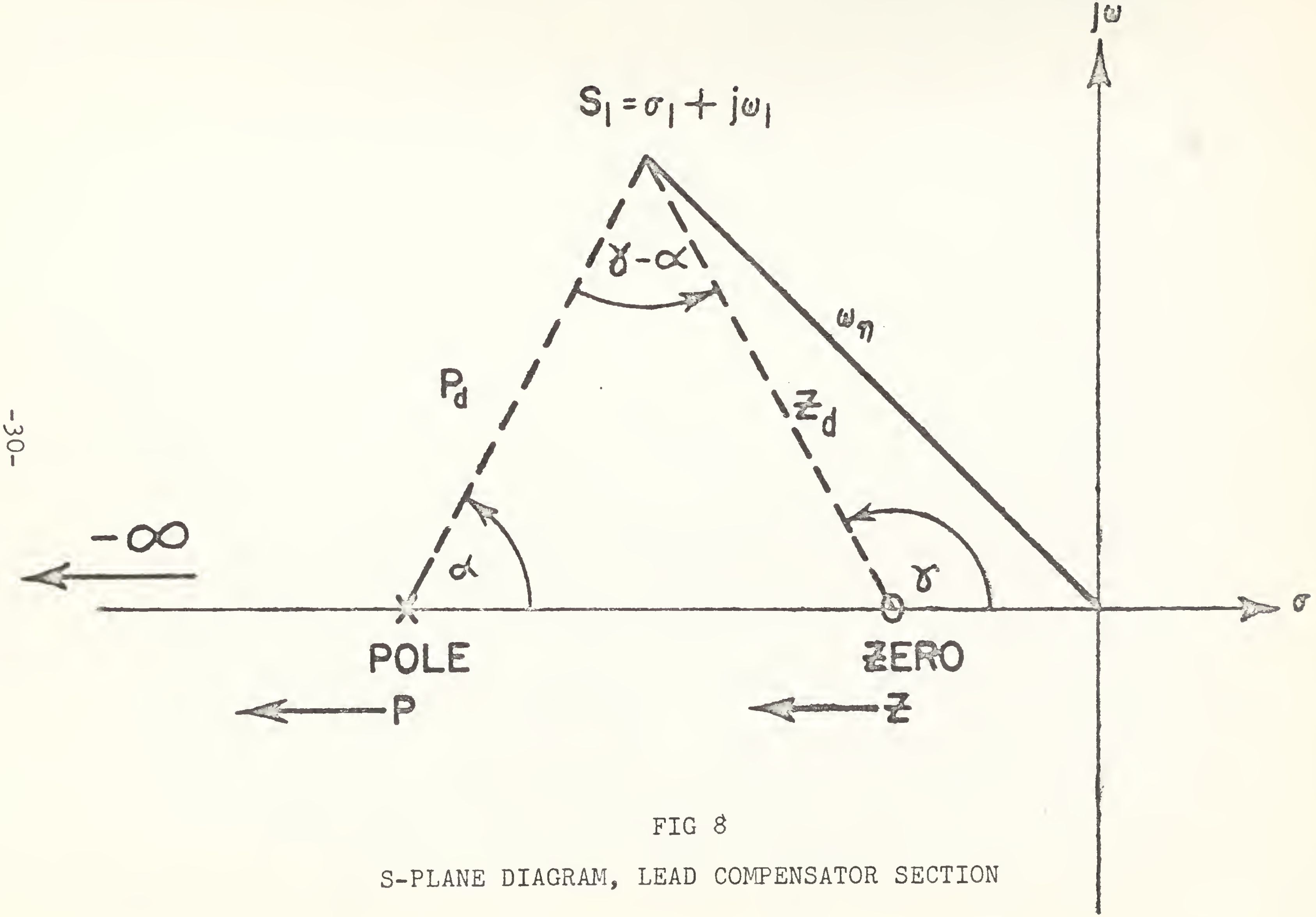


FIG 8

S-PLANE DIAGRAM, LEAD COMPENSATOR SECTION

general case. The desired root location is r . The angle required of a lead compensator to place the root locus through r is the angle $\delta - \alpha$. The zero and pole may be located at any positions on the negative real axis as long as this angle condition is satisfied. The gain factor of the compensator is $\left(\frac{Z}{Zd}\right) \cdot \left(\frac{Pd}{P}\right)$. When the zero is at the origin this gain is zero. If the zero and pole are moved along the negative real axis toward minus infinity in such a way as to maintain the angle requirement, the ratio Pd/P approaches unity, usually without exceeding this value, and the ratio Z/Zd also approaches unity, but frequently exceeds unity, i.e., maximizes, depending on the value of ζ associated with the chosen roots points. As an approximation which provides an upper limit for the gain value, consider the case where the compensator pole is left at minus infinity and the zero is moved along the negative real axis. The gain factor at a selected point is now defined by Z/Zd, since Pd/P = 1.0 . The angle criterion is not satisfied because the pole location is not correct, but the required finite location of the pole can only reduce the gain factor, thus the value obtained is an upper limit for the gain factor of a single section. The result is shown on fig. 9 for ζ = 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, and the figure also shows the phase angle introduced by the zero. These values apply to any system and may be used as a

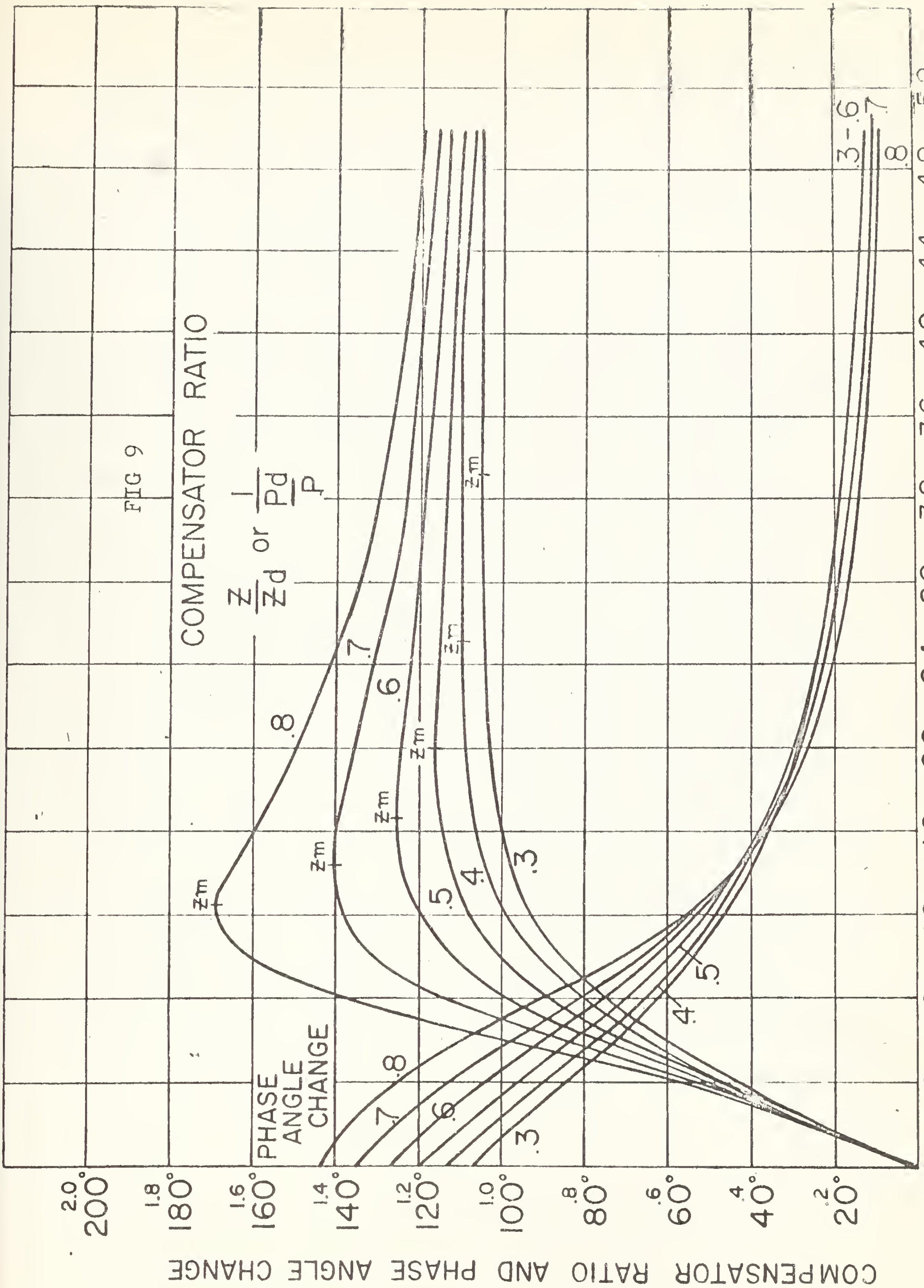


FIG 9

COMPENSATOR RATIO
$\frac{Z}{Z_d}$ or $\frac{1}{\frac{P_d}{P}}$
PHASE ANGLE CHANGE
COMPENSATOR RATIO AND PHASE ANGLE CHANGE
2.0 200°
1.8 180°
1.6 160°
1.4 140°
1.2 120°
1.0 100°
.8 80°
.6 60°
.4 40°
.2 20°
0
.4
.8
1.2
1.6
2.0
2.4
2.8
3.2
3.6
4.0
4.4
4.8
5.2
.3
.4
.5
.6
.7
.8
3-.6
.7
.8
Zm
Zm
Zm
Zm
Zm
Zm

design guide. For a lag compensator similar results may be obtained by placing the zero at minus infinity and moving the pole. The ratio Pd/P is infinite for P at the origin, but drops to a minimum then approaches unity as P approaches infinity.

A single example will demonstrate the use of fig.9 . Suppose r is on the $\zeta = 0.7$ line, and the phase angle at r is 285°. The lead angle required is therefore 105°. Reference to fig. 9 indicates that the zero location required to achieve this lead angle is $0.5\omega n$. At this location, the compensator zero ratio is 0.75 . If this compensator ratio is too small a double lead section might be considered. In this case, each zero must effect one-half of the total lead angle required, or 52.5°. The zero location is found to be at $1.25\omega n$, and each of the two compensators is found to have a zero ratio of 1.39 . Therefore the total maximum compensator ratio is $(1.39)^2 = 1.93$. Likewise three identical sections might be employed. In this case each zero must produce $105°/3 = 35°$ phase lead at r . The zero location is found to be at $1.6\omega n$, and each compensator has a zero ratio of 1.395 . Therefore the total maximum compensator ratio is $(1.395)^3 = 2.72$.

The location of the compensator zero to produce a maximum value for the ratio Z/Zd may be called Zm.

It may be shown that

$$Zm = \omega n/\zeta \qquad (4\text{-}5)$$

$$\frac{Zm}{Zd} = \frac{1}{\sqrt{1-\zeta^2}} \qquad (4\text{-}6)$$

Note that

$$Zd = \sqrt{\omega^2 r + (Zm - \sigma r)^2}$$

then the location of Zm is determined from

$$\frac{d(Zm/Zd)}{d\ Zm} = \left[\frac{d}{dZm}\ \omega^2 r + (Zm - Zd)^2\right]^{-\frac{1}{2}}$$

which evaluates to $Zm = \omega m/\zeta$

It follows that $Zd = \omega_m \sqrt{1-\zeta^2}\ /\zeta$

and $$\frac{Zm}{Zd} = \frac{1}{\sqrt{1-\zeta^2}}$$

Thus the maximum possible gain increase from a single section is easily computed. This gain is never achievable because the compensator cannot be built with the pole at infinity, and the value therefore acts as a guide in that it is an upper limit.

If in a given case where phase lead compensation is required, the pole and zero are both moved along the negative real axis toward minus infinity so as to always satisfy the angle requirement, the zero approaches a limiting position beyond which it cannot be placed if the angle criterion is to be satisfied at r . Calling this limiting location Z_L , it may be noted from fig. 8 that with a required angle $\gamma - \alpha$,

$$\tan(\gamma - \alpha) = \frac{\omega r}{|Z_L| - |\sigma r|} \qquad (4\text{-}7)$$

from which

$$|Z_L| = \frac{\omega r + |\sigma r| \tan(\gamma - \alpha)}{\tan(\gamma - \alpha)} \tag{4-8}$$

Again in practice the zero cannot be located at Z_L, and this value acts as an upper limit in guiding the design. If, for a given case, $Zm < Z_L$, it is possible to compensate with a single section, place the zero at or near Zm, and obtain nearly maximum gain. If $Zm > Z_L$ multiple sections may be required to obtain a gain advantage.

The following simple computations permit ready estimate of the limiting gain values at a selected point r. If Kx is the uncompensated system error coefficient, and Kxc is the compensated system error coefficient, then for one lead section

$$Kxc = Kx / \sqrt{1-\zeta^2} \tag{4-9}$$

for two lead sections

$$Kxc = Kx / (1-\zeta^2) \tag{4-10}$$

for N lead sections

$$Kxc = Kx / (\sqrt{1-\zeta^2})^N \tag{4-11}$$

Note that these error coefficients are upper limits. A similar development is possible for lag compensators. The curves of fig. 9 may be used for this case also.

For a second order system certain relationships can be

established which may be used as figures of merit in selecting compensation for second order systems, and also for higher order systems when the selected roots may be considered dominant. Fig. 10 shows the root locus for a second order type one system, uncompensated. The basic transfer function is $G(s) = Kv/s(s\tau + 1)$, and $Kv = Pdo\ Pd_1 / P_1$, but $Pdo = Pd_1 = \omega n$, and $P_1 = 1/\tau_1$ Thus

$$Kv = \omega n^2 \tau \qquad (4\text{-}12)$$

for the compensated second order system.

When a second order system is compensated by a lead section to produce new complex roots, and the root location is defined by ωnc, ζc, the maximum gain at the new root, as determined from fig. 4 is

$$Kvc = \frac{Pdo\ Pd_1\ Pdm}{P_1\ Pm} \cdot \frac{Zm}{Zdm} = \frac{\omega nc\,\tau\,Pd_1}{\sqrt{1-\zeta^2}} \qquad (4\text{-}13)$$

Pd_1 is less than ωnc, but as a first approximation let $Pd_1 \cong \omega nc$, then

$$Kvc < \frac{\omega nc^2 \tau}{\sqrt{1-\zeta^2}} \qquad (4\text{-}14)$$

If N sections are to be used, then

$$Kvc \quad \frac{\omega nc^2 \tau}{(1-\zeta^2)^{N/2}} \qquad (4\text{-}15)$$

When the required value of Kvc is known, then the minimum number of lead sections is

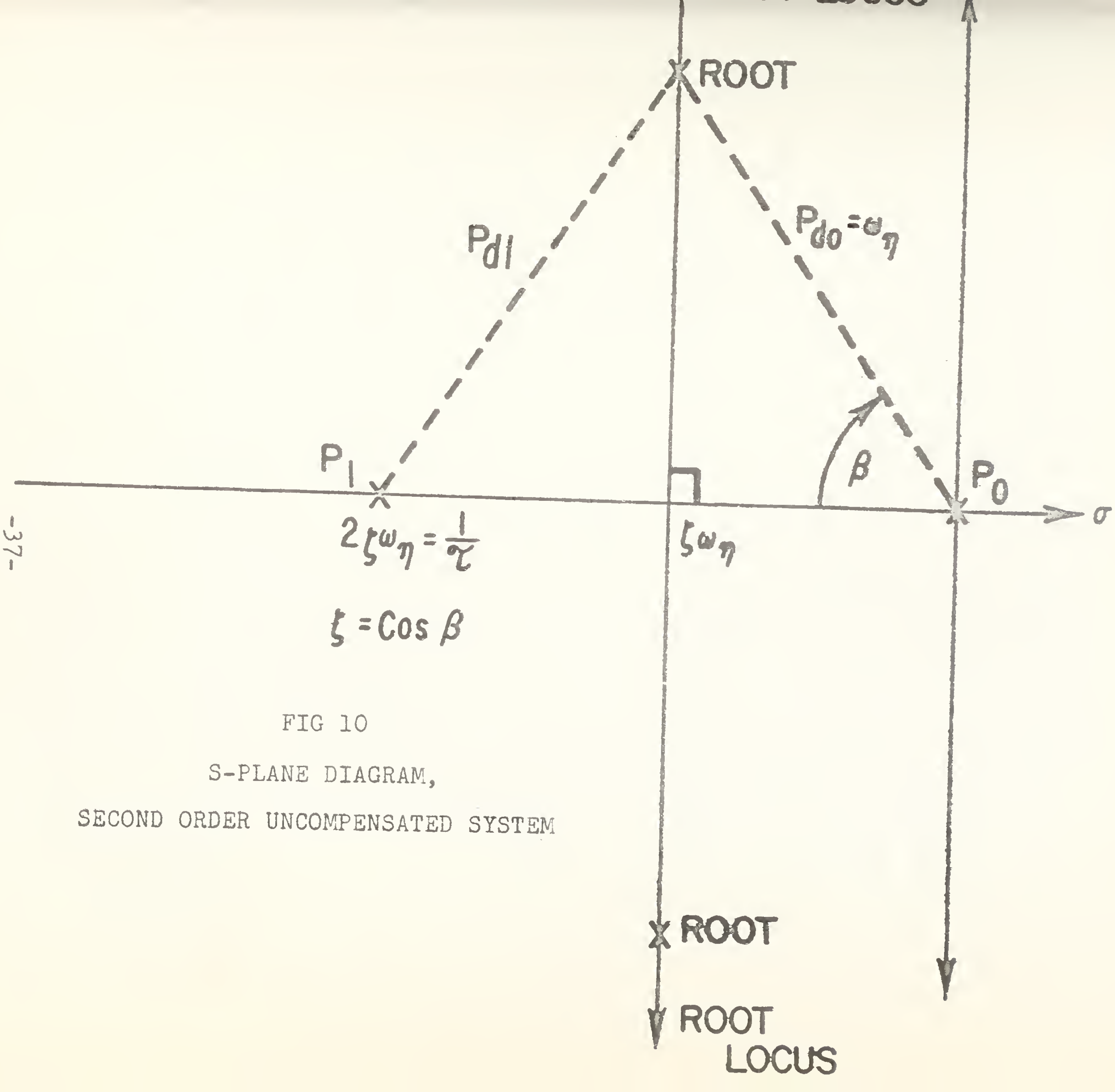


FIG 10

S-PLANE DIAGRAM,

SECOND ORDER UNCOMPENSATED SYSTEM

$$N = \frac{2 \ln (\omega^2 nc\, \tau / Kvc)}{\ln (1 - \zeta^2)} \qquad (4\text{-}16)$$

In practice the number of sections may exceed N appreciably.

If a desired root location can be obtained with only one lead section, it is certainly possible to use more than one section. Thus it becomes of interest to investigate the possibility of obtaining a gain advantage by using multiple sections. As a first step, note that for two identical sections the location of the compensator zero, Z_2, for maximum available gain (which occurs with poles at infinity and without regard to angle requirements) is

$$Z_2 = \frac{\omega n}{\zeta} = Zm \qquad (4\text{-}17)$$

Since $Zd = \sqrt{\omega^2 r + (Z_2 - \sigma r)^2}$

The gain per section is Z_2/Zd, and for two identical sections it is $(Z_2/Zd)^2$.

Maximizing

$$\frac{d(Z_2/Zd)^2}{dZ} = \frac{d}{dZ_2}\left[\omega^2 r + (Z_2 - \sigma r)^2\right]^{-1} = 0$$

This evaluates to

$$Z_2 = \omega^2 m / \sigma r = \omega m / \zeta = Zm$$

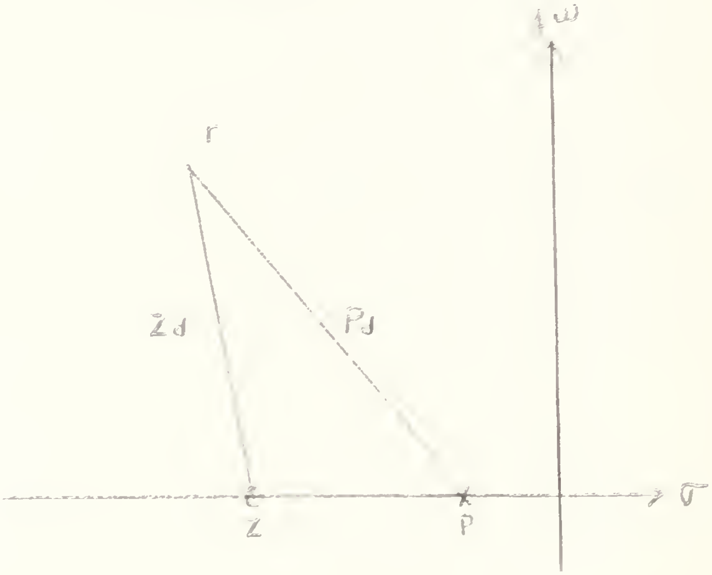
That is, the zero location for N sections is exactly that for a single section. If two non-identical sections are used the theoretical maximum gain cannot be available. This does not prove that a gain advantage necessarily accrues by using multiple sections when one section can do the job, but it is apparent that if two zeros can be

placed at Zm , and the corresponding poles remain far enough out on the negative real axis, then the available gain is increased by a factor $1/\sqrt{1-\zeta^2}$ (due to the second section) which may be an appreciable advantage. In like manner, if the zeros can be kept close to Zm a gain advantage may be available.

LAG SECTIONS

The lag network introduces a net positive angle at r , with the pole of the network to the right of the zero as shown:

FIG 10
LAG NETWORK



Using the same terminology as for the case of a lead network, the Kv of a system will be altered by the factor $\frac{Pd}{P} \cdot \frac{Z}{Zd}$

In its usual configuration, the lag network is essentially a dipole near the origin, with $Z < 1$, $P < 1$. In this

case $Zd \cong Pd$, and the compensator ratio of a lag network is Z/P .There will be little phase angle change at r produced by a dipole, since the angle contribution of Z and P are approximately equal. Thus the value of Kv may be changed easily by altering the ratio of Z and P .

The usual objections to the employment of lag networks must be considered.The use of a dipole near the origin generally will produce an additional root near the origin with an attendant long time constant. This root may have an undesirable effect upon transient performance. In addition, general engineering considerations of noise must be remembered. Generally lag networks are to be avoided; however, because they do permit the realization of high steady state accuracy, their use may be necessary in those applications where passive networks are desired.

For example, suppose an increase in Kv by a factor of 5 is required. From the preceding section it is apparent that this gain cannot readily be achieved solely through the use of lead sections. In this case by adopting a ratio of $Z/P = 5$ for a lag network, the necessary Kv can be obtained. $Z = .20$, $P = .04$ is a possible combination. Actual values can be easily adjusted to conform to available components.

LEAD-LAG NETWORKS

In the use of lead-lag networks, sucessive application of these same principles developed previously may be employed. A lead network is placed in a location such that the phase angle at r is 180°, and a lag network added to produce the required Kv . The lag network generally will have little effect upon the angle at r ; however, the lead network may have to be altered slightly after the introduction of the lag network to again produce a phase angle of exactly 180°. This final modification of the lead network should have little effect on the final value of Kv .

COMPENSATION PROCEDURE

Compensation may be designed using these relationships by:

1. Selecting r , evaluating $\underline{|G(r)}$, Kx. Estimate the number of sections required to satisfy the angle requirement, and compare Kx with the value required by specifications. Since the gain increase available from a lead section is small unless $\zeta > 0.7$ (see fig. 9) Kx must be near the specified value or lag compensation is required. If required insert a lag compensator to adjust Kx to an acceptable value and proceed with the design of a lead compensator . Note that this is a major advantage of this method; it defines the need for lag compensation at an early point.

2. To design the lead compensator, estimate Zm , using Fig. [illegible], and [illegible] , place one or more zeros as close to Zm as permitted by [illegible] , determine the corresponding pole locations and check Kxc. If the preliminary calculations were interpreted intelligently Kxc should be at or above the specified value, otherwise a second trial may be required.

3. Check the location of the roots introduced by the compensators, and estimate their effect on the dynamic response. If necessary readjust the compensator design.

GENERAL COMMENTS

The principles discussed in this chapter make possible rapid solutions to problems involving servo systems which may be compensated by passive networks in the direct path. In addition the principles are applicable to other forms of compensators as well. For example, derivative feedback generally produces a zero in the open-loop pole-zero configuration. Placement of this zero at Zm will have the same effect upon the system Kv as will the zero of a phase lead compensator in the direct path.

The theoretical limits on Kv (or on Kp, Ka, Etc.) may be approached closely in actual design. This is due to the fact that lead filters with an attenuation ratio

of 10:1 produce an increase in Kv which is very close to the theoretical limit. To achieve this, the zero position is placed slightly closer than (but very near to) the theoretical position for Zm . With an attenuation ratio of 10:1, the pole will be far enough out on the negative real axis so that its contribution to Kv and phase angle is relatively small. Thus the designer need only calculate the value of Kv contribution of the poles and zeros of the uncompensated system and the limiting value of a lead (or double lead, etc.) section to determine if compensation by lead networks alone is feasible.

The following relationships summarize the principal concepts :

1. $Zm = \frac{1}{\zeta}\omega n$ for a lead compensator

2. $\frac{Zm}{Zd} = \frac{1}{\sqrt{1-\zeta^2}}$, the maximum theorectical gain for a lead compensator section.

3. Increase in Kv produced by a lag compensator = Z/P provided that the lag network is essentially a dipole near the origin.

4. For a second order system with one lead compensator

$$Kvc < \frac{\omega^2 n \tau}{\sqrt{1-\zeta^2}}$$, where $\tau = 2\zeta\omega n$

5. For a second order system with two lead compensator sections

$$Kvc < \omega^2 n \tau/(1-\zeta^2)$$

CHAPTER V. Selected Examples.

The following examples are given to illustrate some of the concepts which have been developed in preceding chapters.

EXAMPLE V-1

Given: $G(s) = \frac{588,000}{s(s + 4)(s + 600)}$

Requirements: Kv not be reduced. Desired root location to be such that $\sigma r = 15$, $0.5 \leq \zeta \leq 0.7$ (see fig. 11a) Lag networks to be avoided. Attenuation ratios of lead networks must not 10.

Solution: Because the effect of the pole at s=600 is negligible, the system may be considered second order. The uncompensated system has a Kv :

$$Kv = \frac{K}{P_1 P_2} = \frac{588,000}{(4)(600)} = 245$$

At $s_1 = \sigma_1 + j\omega_1$, for $\zeta = 0.7$

$$Kvu = \frac{(Pdo)(Pd_1)}{P_1} = \frac{15\sqrt{2} \quad 18.7}{4}$$

since we are neglecting the pole at s = -600.

Or,

$$Kvu < \omega_{nc_1}^2 \tau = \frac{(15\sqrt{2})^2}{4} = 112$$

With one lead section, from equation 4-14,

$$Kvc < \frac{\omega_{nc_1}^2 \tau}{\sqrt{1-\zeta^2}} = \frac{112}{\sqrt{1 - .49}} = 157$$

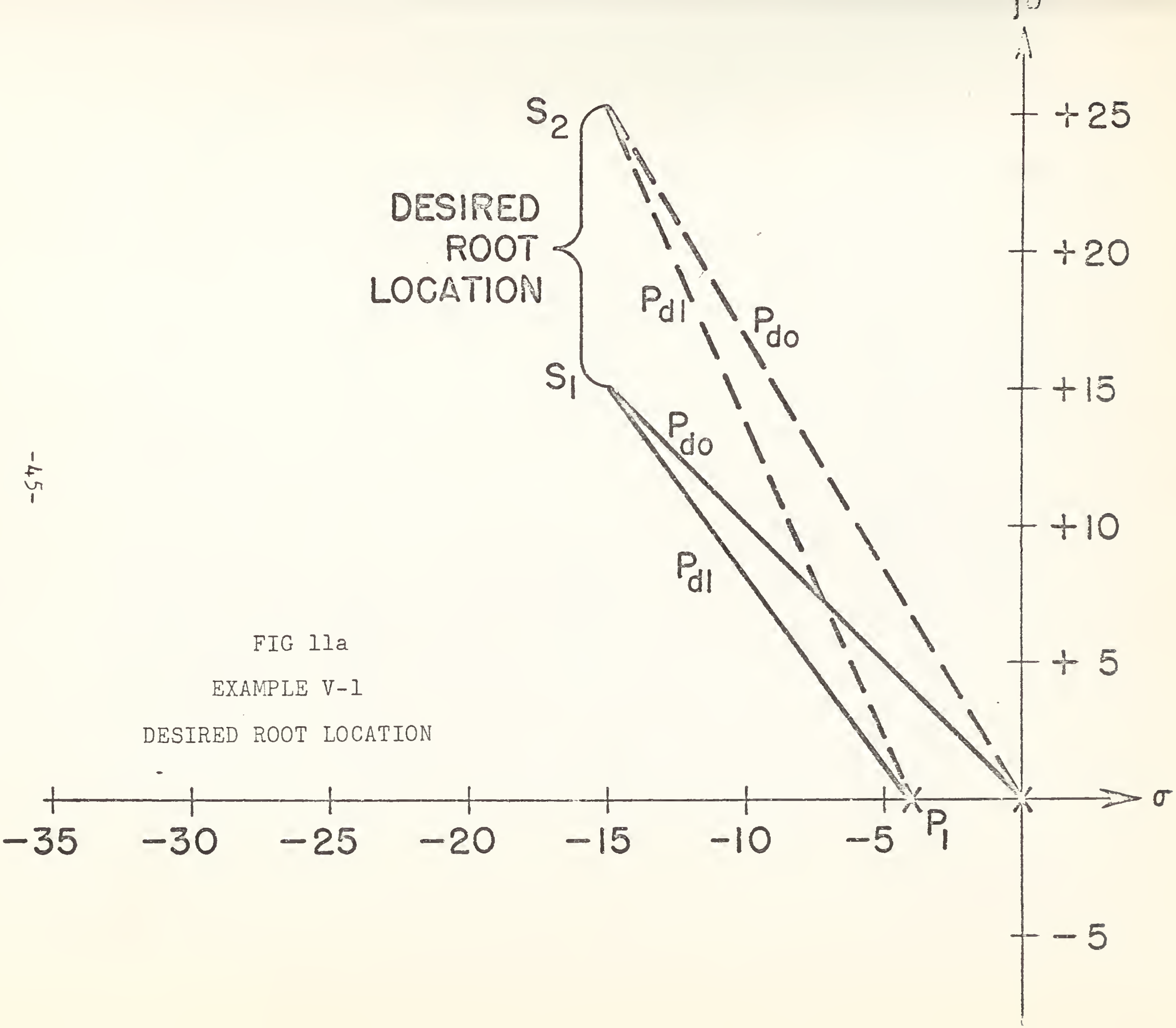


FIG 11a

EXAMPLE V-1

DESIRED ROOT LOCATION

With two lead sections,

$$Kvc < \frac{\omega^2 nc_1 \tau}{1 - \zeta^2} = \frac{112}{1 - .49} = 220$$

Since neither of these two conditions will meet requirements, examine the root $s_2 = \sigma_2 + j\omega_2$, for $\zeta = 0.5$

$$Kvu < \omega^2 nc \; \tau = \frac{(30)^2}{4} = 225$$

With one lead section,

$$Kvc < \frac{\omega^2 nc_2 \tau}{\sqrt{1 - \zeta^2}} = \frac{225}{\sqrt{1 - .25}} = 260$$

With two lead sections,

$$Kvc < \frac{\omega nc_2 \tau}{1 - \zeta^2} = \frac{225}{1 - .25} = 300$$

The value of Kvc shown by one section would appear to be sufficient to meet specifications but is probably too close to the required value to be realized. The value for two sections appears to be more practical. From equation 4-5,

$$Zm = \omega n/\zeta = 30/.5 = 60$$

Furthermore, the phase angle at $s_2 = \underline{/G(s = s_2)} = 235°$. Since two sections are to be used, each section must produce $\frac{235 - 180}{2} = 27.5°$ lead. Using fig. 9 , this occurs when the Z location for the compensator is at $Z = 2.2\,\omega nc = 2.2(30) = 66.0$. Also from fig. 9 , a zero at $Z = 2.2\,\omega nc$ produces a Z/Zd ratio of 1.14 . For two sections,

$$\left(\frac{Z}{Zd}\right)^2 = (1.14)^2 = 1.29$$

From this we can calculate the maximum theoretical Kvc:

$$Kvc = Kvu(1.29) = \frac{Pdo\ Pd_1\,(1.29)}{P_1}$$

$$= \frac{(30)(26.7)(1.29)}{4} = 256$$

Actual Kv after compensation will be somewhat less than this value because of the finite poles. Because the poles are finite, the zero of the compensator must be placed closer to the origin than $Z = 66.0$. For a trial point, try $Z = Zm = 60.0$. By graphical means, the pole location necessary to produce $\underline{/G(s = s_2)}$ is 640. This produces a phase lead attenuation of $640/60 = 10.67$, which is slightly greater than specifications permit. A second try with the zeros at $s = 59.0$ yields a pole position of 570, with a resultant attenuation ratio of $570/59 = 9.68$. Including

the pole at $s = -600$, the final Kv is

$$Kvc = \frac{(Pdo)(Pd_1)(Pd_2)}{(P_1)\ (P_2)} \left(\frac{Zc}{Zdc}\right)^2 \left(\frac{Pdc}{Pc}\right)^2$$

$$= \frac{(30)(267)(586)(59)^2\ (556)^2}{(4)(600)\quad (51)^2\ (570)^2} = 250$$

$$Gc(s) = \frac{(56.0)\cdot 10^6}{s(s+4)(s+600)} \cdot \left(\frac{s+59}{s+570}\right)^2$$

The following points should be noted:

1. The final Kvc = 250 compares to the value of 300 given by the second order formula, equation 4-15.

2. The Kvc = 250 is within 2.4% of the value given by fig. 9 . With attenuation ratios equal to 10, accuracies within 5% can be expected. Note that this accuracy was achieved even though the pole at $s = -600$ was neglected until the final calculation.

3. Even though greater compensator ratios $(1/\sqrt{1-\zeta^2})^N$ may be achieved for larger ζ locations,

$$Kxc = Kxu/\left(\sqrt{1-\zeta^2}\right)^N$$

will generally be greater for smaller ζ locations, since the quantity Kxu will dominate.

The root locus for the compensated system is shown on fig. 11b.

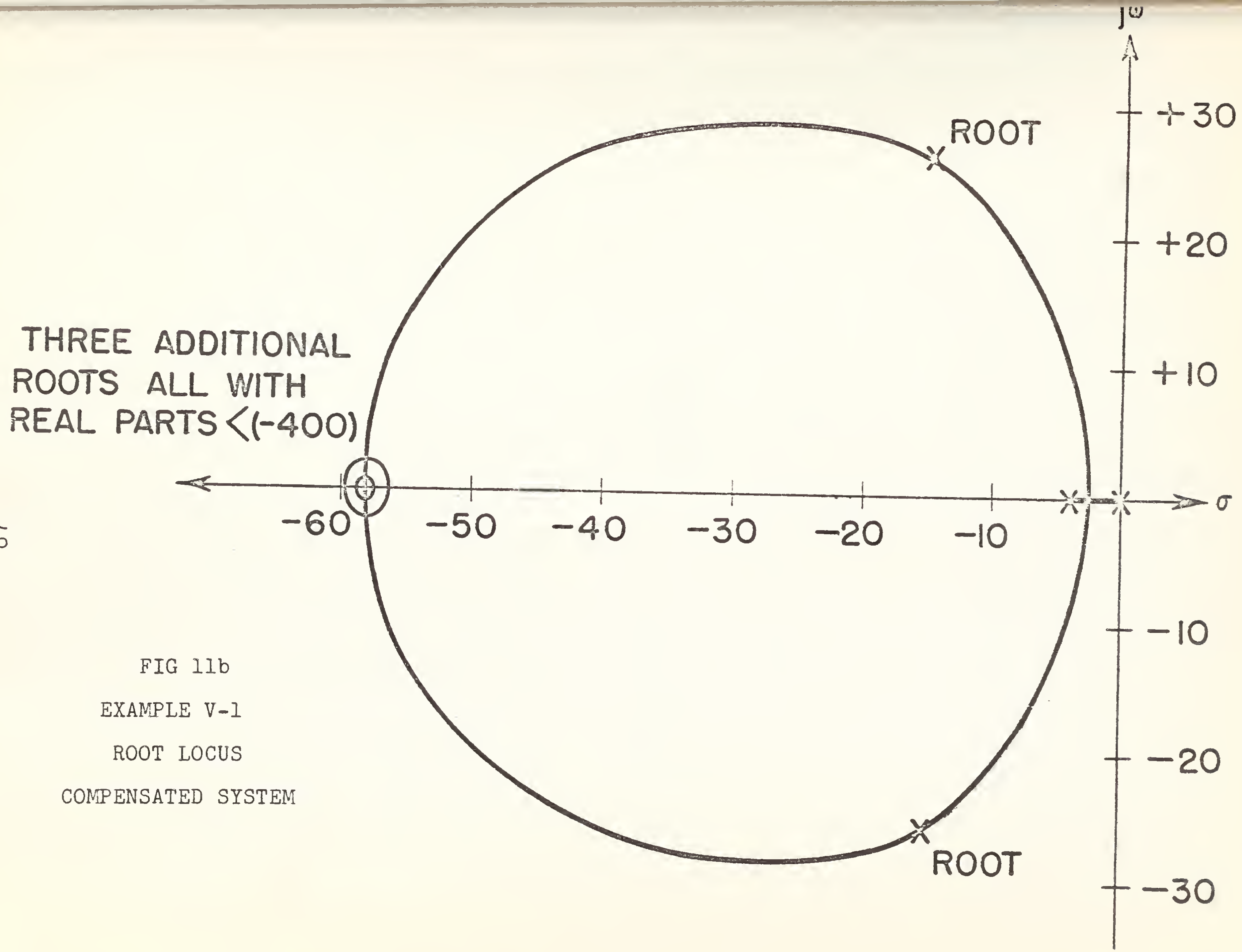


FIG 11b

EXAMPLE V-1

ROOT LOCUS

COMPENSATED SYSTEM

EXAMPLE V-2

Given: $G(s) = \dfrac{7000}{s(s + 10)(s + 35)}$

Requirements: Kv is not to be reduced. Desired root location is $r = -15 \pm j\,15$. Lag networks are to be avoided if possible, and the attenuation of lead networks must not exceed 10.

Solution:

$$Kv = 7000/(10)(35) = 20$$

$$\underline{\angle G(-15 + j\,15)} = 280°$$

Therefore 100 degrees phase lead is needed.

$$Kvu = \frac{Pdo\ Pd_1\ Pd_2}{P_1\ P_2} = 24$$

Therefore the gain ratio of a lead compensator must not be less than 20/24 = 0.834 .

From fig. 9 , for $\zeta = 0.7$, one section will not work, since both the angle and compensator ratio requirements cannot be satisfied simultaneously. For two identical sections each section must produce 50 degrees lead with a gain per section of $\sqrt{.834} = 0.915$. From fig. 9 , the zero may be placed at $1.25\ \omega_n$, with gain ratio of 1.39 per section. Therefore two sections should be adequate. To keep the gain at about Kv = 20, try a zero at $0.6\ \omega_n$. The zero then produces 90° lead. To reduce the section angle to 50°, pole must be placed in the position to produce 40° lag. From fig. 9 , the pole

goes at $1.55\,\omega n$. This does not work, however, because the Kv requirement is not satisfied. Try the zero at $0.95\,\omega n$ ($\sigma = -20$), the pole goes at $2.7\,\omega n$ ($\sigma = -57$), section gain ratio including the pole is 0.994 and $Kvc = 24(.994)^2 = 22.7$. Finally, with zero at $\sigma = -19$, the pole at $\sigma = -47$, the gain becomes $Kv = 20.02$, and the compensated transfer function is

$$Gc(s) = \frac{43{,}300(s + 19)^2}{s(s + 10)(s + 35)(s + 47)^2}$$

The compensated root locus is shown on fig. 12a. The compensator has introduced two complex conjugate roots very near to the desired roots. The transient response is acceptable, however, as indicated on fig. 12b.

EXAMPLE V-3

Given: G(s) as in Example V-2

Requirements: roots at $-15 \pm j\,15$, no lag compensators, only two lead sections permitted, with attenuation ratios less than 10. Design lead sections for maximum Kv.

Solution: As in Example V-2, 50 degrees lead per section is required. From fig. 9, maximum zero coordinate is at $1.13\,\omega n$ ($\sigma = -27.4$) but this violates the attenuation restriction. Try zero at $\sigma = -26$. Pole goes at $\sigma = -220$. Attenuation ratio is 220/26 = 8.5, which is acceptable and should give nearly maximum gain.

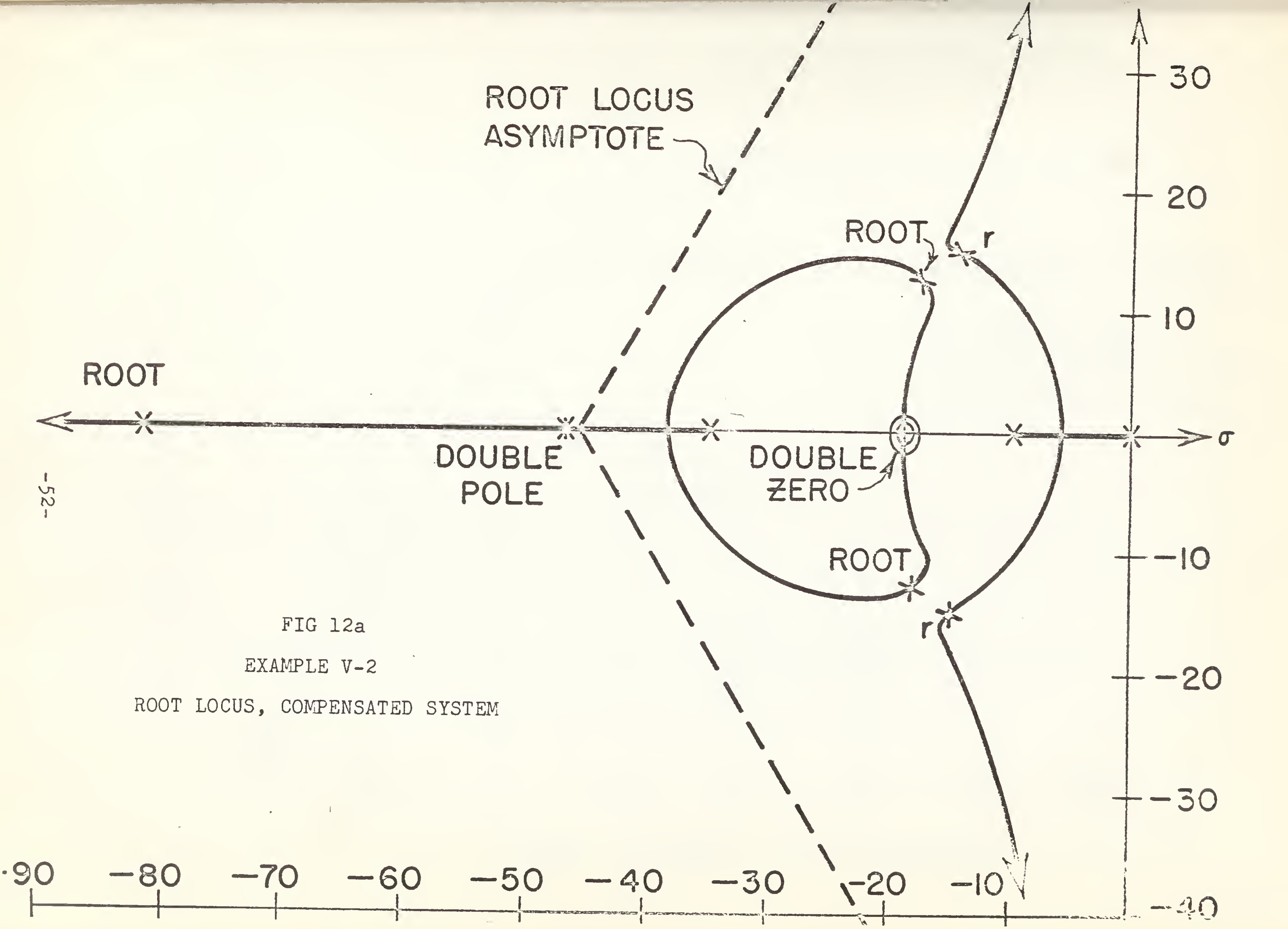


FIG 12a

EXAMPLE V-2

ROOT LOCUS, COMPENSATED SYSTEM

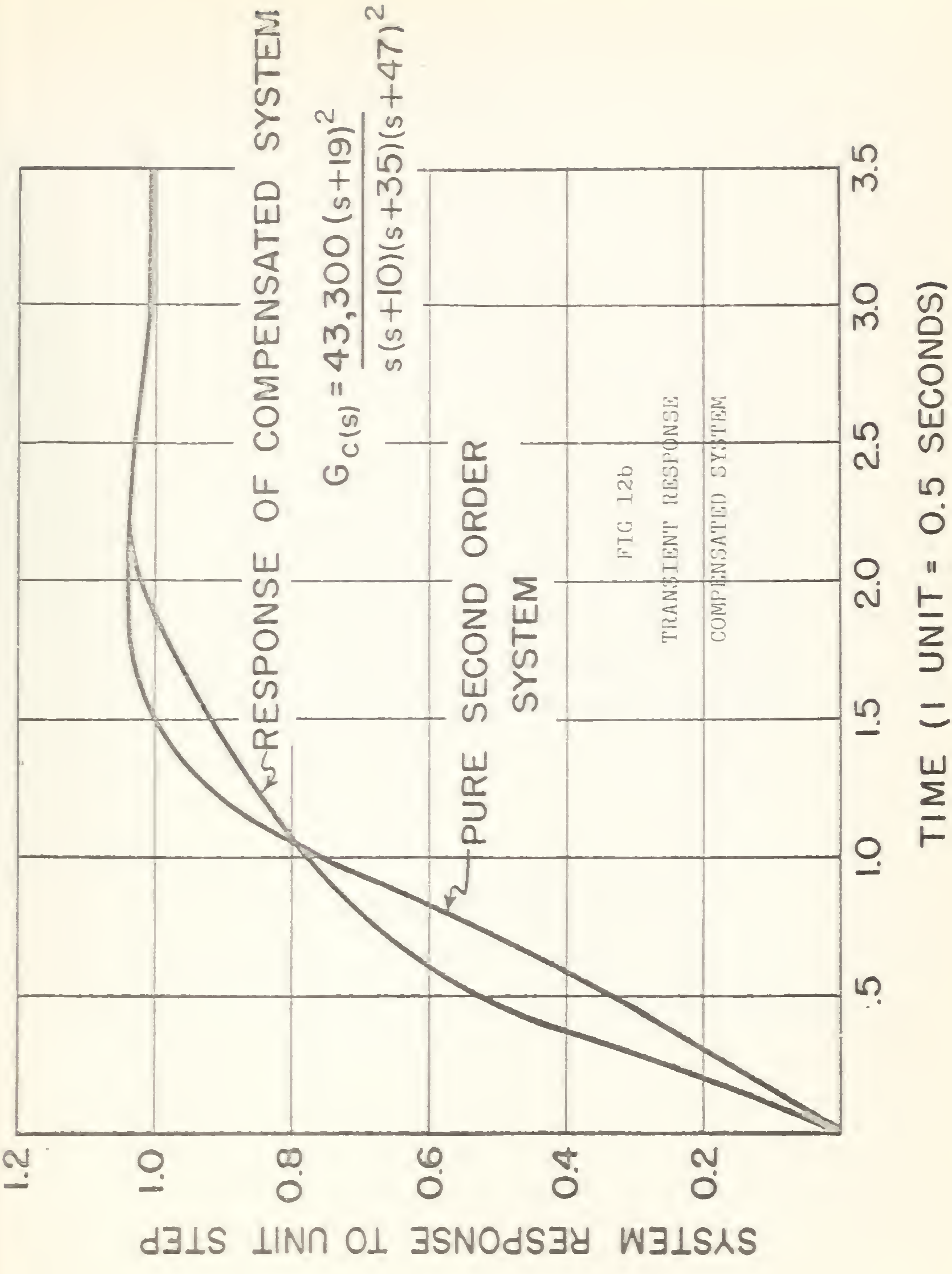


FIG 12b
TRANSIENT RESPONSE
COMPENSATED SYSTEM

The compensated system transfer function is

$$G_c(s) = \frac{989{,}000(s + 26)^2}{s(s + 10)(s + 35)(s + 220)^2}$$

$$K_{vc} = 39.5$$

The maximum theoretical value is

$$K_{vc} = K_{vu}/(1 - \zeta^2) = 24/.51 = 47$$

A slight gain increase is possible in the compensated system by further manipulation until the attenuation ratio is closer to the maximum permitted value, 10 , but the further gain increase would be small. It is interesting to note that the gain obtained is within 15% of the theoretical maximum. The compensated root locus is shown in fig. 13 .

EXAMPLE V-4

Given: $G(s) = \frac{150{,}000}{s(s + 2)(s + 10)(s + 15)}$

Requirements: Kv not to be reduced. Desired root location $\sigma_r = -10$, $\zeta \geq 0.5$. It is desired to minimize the number of compensating sections.

Solution:

$$K_v = \frac{K}{P_1 P_2 P_3} = \frac{150{,}000}{(2)(10)(15)} = 500$$

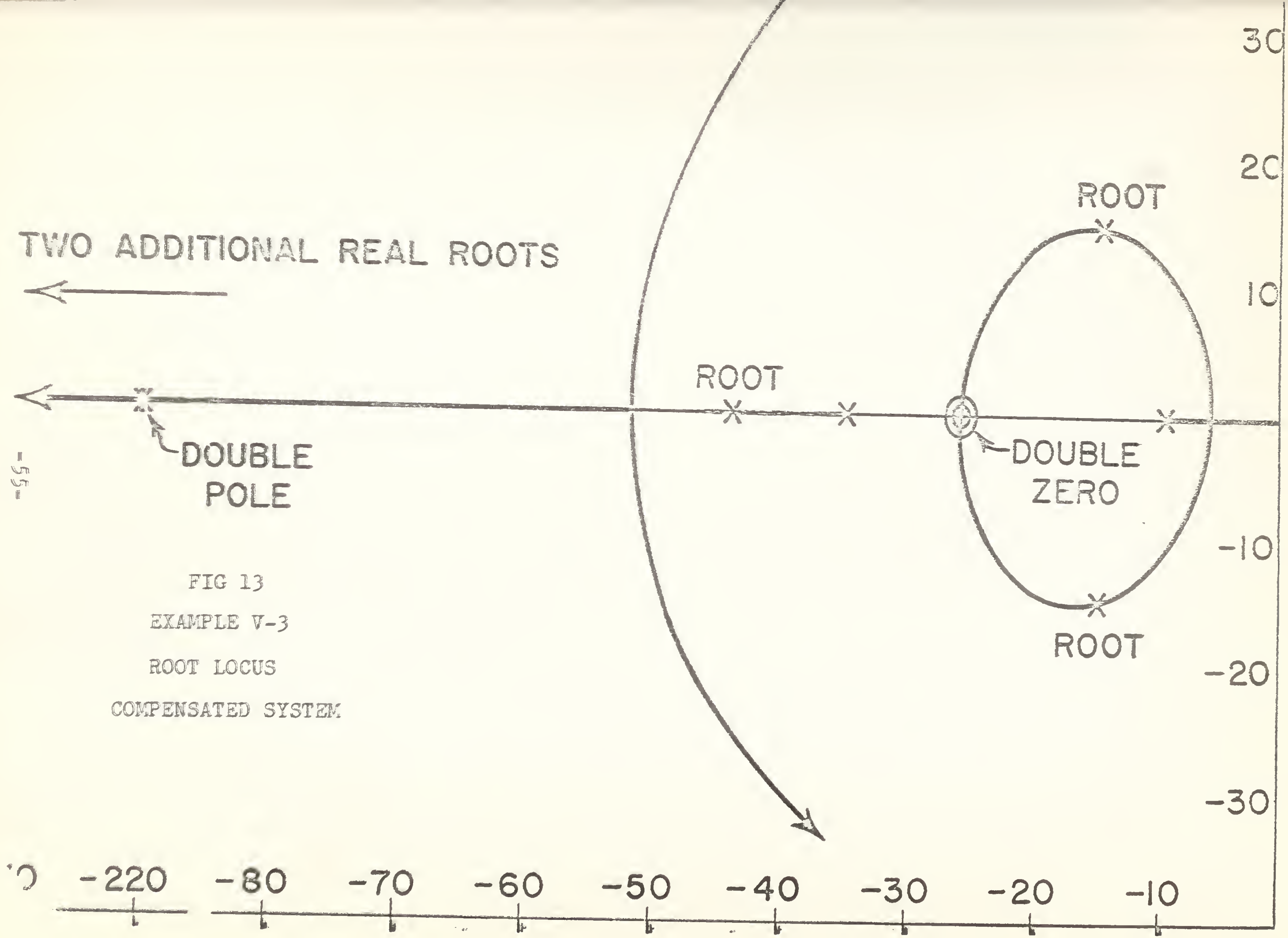


FIG 13

EXAMPLE V-3

ROOT LOCUS

COMPENSATED SYSTEM

With the root at $\sigma_r = -10$ and $\zeta = 0.5$

$$K_{vu} = \frac{P_{do}\ P_{d1}\ P_{d2}\ P_{d3}}{P_1\ \ P_2\ \ P_3} = \frac{(20)(19)(17.7)(18.5)}{(2)(10)(5)}$$

$$= 415$$

The phase angle at the root is 400°. If s_1 is to be placed on the 180° phase angle locus, 220° of phase lead compensation must be used. Because of the high K_v requirement, root locations at positions where the damping ratio ζ is greater than 0.5 will not lend themselves to an easier solution. Therefore consider only this one location; namely, that where $\zeta = 0.5$.

If two sections of phase lead networks are used, each must produce 220/2 = 110° of phase lead. Reference to fig. 9 indicates that this occurs when the zeros are placed at $s = .175\,\omega_n$ (for $\zeta = 0.5$). However, if lead attenuation ratios are not to exceed 10, then the poles of the lead networks must be placed at $10(.175\,\omega_n) = 1.75\,\omega_n$. Reference to fig. 9 shows that a pole at this location will produce a phase lag of 33°. The resultant net angle per section would be 110° - 33° = 77° Thus it is quickly apparent that this method would be unsatisfactory.

If three sections of phase lead compensation are used, each must produce 220/3 = 73.3°. If the zeros are placed

at a point to produce 90° lead, then the poles can be located at that position to produce $(90 - 73.3)° = 16.7°$ lag. From fig. 9, the zero position, for $\zeta = 0.5$, is at $s = -.5\,\omega n = -.5(20) = -10$; the pole position is at $s = 3.5\omega n = -3.5(20) = -70$.

$$\text{The lead compensator ratio} = \frac{Z}{Zd} \cdot \frac{Pd}{P} = (.6)\frac{1}{1.15} = .522$$

per section. Since three sections are to be used, this ratio must be cubed, and $(.522)^3 = .142$

$$Kvc = Kvu(.142) = (415)(.142) = 58.9$$

The Kv requirement is still unsatisfied. An additional section, either lead or lag, must be added in order to increase Kvc to the desired value of 500. The question arises whether this may be accomplished by a fourth lead section or whether a lag section should be used. The answer may be quickly determined. If four sections were to be used, each must produce 220/4 = 55° lead. From fig. 9 , this occurs when the zero is placed at $s = -1.1\,\omega n$, with a resultant $\frac{Z}{Zd}$ ratio of 1.03 . Under these circumstances, Kvc would be given by

$$Kvc = Kvu(Z/Zd)^4 = (415)(1.03)^4 = 467$$

and this does not account for the attenuation due to the finite pole locations. Clearly a lag compensator

must be used if the compensation is to be accomplished with four sections.

From the above, with three lead sections utilized to produce the proper phase angle at s_1, $K_{vc} = 58.9$. Therefore the lag compensator must produce a gain of $500/58.9 = 8.5$. A value of 9.0 might be used to correct for any errors in calculation. To produce this gain, it is simply necessary to design a dipole with

$$\frac{Z}{P} \approx 9.0$$

Select arbitrarily a pole position of $s = -0.1$ a zero position of $s = -0.9$. The zero values of the lead compensator are placed at the determined position of $s = -10$, and the pole position of the lead compensators are placed in the necessary position to produce the 180° phase angle at s_1. This may be done graphically (using the method described in Chapter III) or analytically, but note that the final position will be slightly different than the position $s = -70$ previously calculated because of the slightly different phase angle at the root location caused by the lag compensator. This position is determined to be $s = -65$.

The final compensation is, therefore

$$Gc(s) = \frac{4.29\cdot 10^6}{s(s+2)(s+10)(s+15)} \cdot \left(\frac{s+10}{s+65}\right)^3 \cdot \left(\frac{s+0.9}{s+0.1}\right)$$

$$Kvc = \frac{4.29\cdot 10^6(10^3)(9\cdot 10^{-1})}{(2)(10)(15)(65^3)(10^{-1})} = 578$$

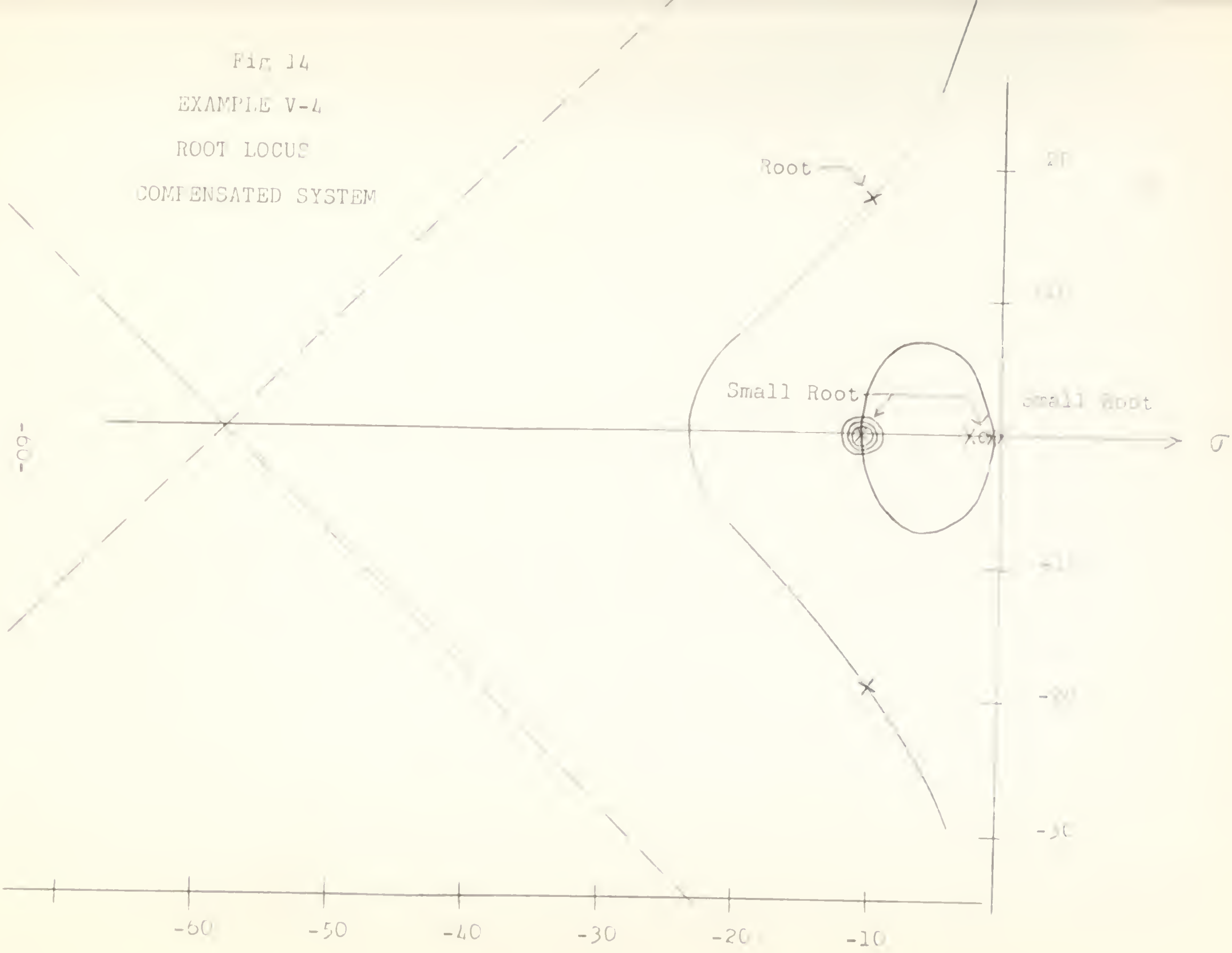
The compensated root locus is shown on fig. 14 .

Fig 14

EXAMPLE V-4

ROOT LOCUS

COMPENSATED SYSTEM



CHAPTER VI. Conclusions.

1. This paper shows a method for computing effects of compensator poles and zeros on the transfer function gain and phase at any selected point in the s-plane.

2. Curves summarizing these effects have been presented for the case of poles and zeros on the negative real axis, with the selected points on constant ζ contours in the left half plane.

3. A design technique is developed which permits simultaneous consideration of root location specifications, and restriction on the type and number of compensator sections.

4. Ease in application of the technique and accuracy of results have been demonstrated.

BIBLIOGRAPHY

1. Thaler, G. J., Elements of Servomechanism Theory, McGraw-Hill Book Co., Inc., 1956.

2. Evans, W. R., Control System Dynamics, McGraw-Hill Book Co., Inc., 1954.

3. Savant, Jr., C. J., Basic Feedback Control System Design, McGraw-Hill Book Co., 1958.

4. Thaler, G. J. and Brown, R. G., Analysis and Design of Feedback Control Systems, McGraw-Hill Book Co., 1960.

5. Ross, E. R., and Warren, T. C., and Thaler, G. J., Design of Servo Compensation Based on the Root Locus Method, AIEE Paper 60-779.

6. Chu, Yaohan, Synthesis of Feedback Control Systems by Phase-Angle Loci, TRANS AIEE Pt. II, November, 1952.

7. Truxal, J. G., Automatic Feedback Control System Synthesis, McGraw-Hill Book Co., Inc., 1955.

thesP585
An exact method of servomechanism compen
3 2768 000 99277 0
DUDLEY KNOX LIBRARY